1 The fundamental matrix (Green function)

- Formulate for Markov chains with an absorbing boundary. (Applications to other chains can be made by suitable sets of states to make them absorbing.)

Suppose the state space \( S = I \cup B \) satisfies the following regularity condition:

- Finite number total number of internal states \( I \).
- Finite or countably infinite set of boundary states \( B \): each \( b \in B \) is absorbing, meaning that \( P(b, b) = 1 \).
- Starting at any \( i \in I \), there is some path to the boundary, say \( i \rightarrow j_1 \rightarrow j_2 \rightarrow \cdots \rightarrow j_n \in B \) with \( P(i, j_1)P(j_1, j_2)\cdots P(j_{n-1}, j_n) > 0 \).

It’s easy to see that \( \mathbb{P}_i(T_B < \infty) = 1 \) for all \( i \in I \): say \( |I| = N < \infty \), no matter what the starting state \( i \), within \( N + 1 \) steps, by avoiding any repeat, there must be a path as above from \( i \) to some \( j_n \in B \) in \( n \leq N \) steps. So \( \mathbb{P}_i(T_B \leq N) > 0 \) for each \( i \in I \). Because \( I \) is finite, this implies

\[
\epsilon := \min_{i \in I} \mathbb{P}_i(T_B \leq N) > 0.
\]

Then

\[
\mathbb{P}_i(T_B > N) \leq 1 - \epsilon
\]
\[
\mathbb{P}_i(T_B > 2N) \leq (1 - \epsilon)^2 \quad \text{(Condition on } X_N \text{)}
\]

\[
\vdots
\]
\[
\mathbb{P}_i(T_B > kN) \leq (1 - \epsilon)^k \quad \downarrow \text{0 as } k \uparrow \infty
\]

\[
\implies \mathbb{P}_i(T_B < \infty) = 1
\]

That is, if \( T_B = \infty \), then \( T_B > KN \) for arbitrary \( K \), so

\[
0 \leq \mathbb{P}_i(T_B = \infty) \leq \mathbb{P}_i(T_B > KN) \leq (1 - \epsilon)^K \rightarrow 0 \text{ as } K \rightarrow \infty.
\]

Therefore \( \mathbb{P}_i(T_B = \infty) = 0 \). Also, \( T_B \) has finite expectation:

\[
\mathbb{E}_i(T_B) \leq N \sum_{K=0}^\infty \mathbb{P}_i(T_B > KN) \leq \sum_{K=0}^\infty (1 - \epsilon)^K = \epsilon^{-1}
\]
• (Text Page 169) Assume the structure of the transition matrix $P$ is as follows:

$$
P = \begin{pmatrix}
I & Q \\
B & R
\end{pmatrix}
$$

$Q_{ij} = P_{ij}$ for $i, j \in I$ and $R_{ij} = P_{ij}$ for $i \in I, j \in B$. Note that $Q$ is a square matrix, and $I$ denotes an identity matrix, in this diagram indexed by $b \in B$, and later indexed by $i \in I$. Matrix $R$ indexed by $I \times B$ is typically not square.

• Key observation: Look at the mean number of hits on $j$ before hitting $B$ starting at some different $i \in I$. (Always count a hit at time 0)

First: case $j \in I$

$$
\mathbb{E}_i \left[ \sum_{n=0}^{\infty} 1(X_n = j) \right] = \sum_{n=0}^{\infty} P^n(i, j) = \sum_{n=0}^{\infty} P^n(i, j)
$$

Observe that for $i, j \in I$, $W_{ij} := \mathbb{E}_i [\sum_{n=0}^{\infty} 1(X_n = j)] = \mathbb{E}_i [\sum_{n=0}^{\infty} P^n(T_B > n)] = \mathbb{E}_i (T_B) \leq \epsilon^{-1} < \infty$

so $W := (W_{ij}, i, j \in I)$ is an $I \times I$ matrix with all entries finite.

Claim: $P^n(i, j) = Q^n(i, j)$ for all $i, j \in I$.

$W_{ij} = \sum_{n=0}^{\infty} Q^n_{ij}$ for $i, j \in I$ is a finite matrix. Consider the following analogy:

**Real numbers:** $w = \sum_{n=0}^{\infty} q^n = 1 + q + q^2 + \cdots = (1 - q)^{-1}$

**Matrices:** $W = \sum_{n=0}^{\infty} Q^n = I + Q + Q^2 + \cdots = (I - Q)^{-1}$

How do we know $I - Q$ is invertible? Let us prove that $W$ really is an inverse of $I - Q$. Need to show: $W(I - Q) = I$, that is, $W - WQ = I$ or $W = I + WQ$.

Check:

$$
W = \sum_{n=0}^{\infty} Q^n = I + Q + Q^2 + \cdots
$$

$$
WQ = \sum_{n=1}^{\infty} Q^n = Q + Q^2 + Q^3 + \cdots
$$

$$
\Rightarrow I + WQ = W
$$

• Summary: In this setting, matrix $W$ is called the **Fundamental Matrix**, also the **Green Matrix (or Function)**. We find that $W = (I - Q)^{-1}$. The meaning of entries of $W$ is that $W_{ij}$ is the mean number of hits of state $j \in I$, starting from state $i \in I$. 

Final points:

1. \( W \) and \( R \) encode all the hitting probabilities:

\[
P_i(X_{TB} = b) = \sum_{n=1}^{\infty} P_i(T_B = n, X_n = b) = \sum_{n=1}^{\infty} \sum_{j \in I} P_i(X_{n-1} = j, X_n = b)
\]

\[
= \sum_{n=1}^{\infty} \sum_{j \in I} \sum_{n=1}^{\infty} Q_{ij}^{n-1} P_{jb}
\]

\[
= \sum_{j \in I} W_{ij} P_{jb} = (WR)_{ib} \quad \text{since } R = P|_{I \times B}
\]

2. From \( Q^n \) and \( R \) you pick up distribution of \( T_B \):

\[
P_i(T_B = n) = \sum_{b \in B} P_i(T_B = n, X_n = b) = \sum_{b \in B} \sum_{j \in I} Q_{ij}^{n-1} R_{jb}
\]

\[
= \sum_{j \in I} W_{ij} P_{jb} = (Q^{n-1}R)_i
\]

3. Exercise: \( Q^n \) and \( R \) give you the joint distribution of \( T_B \) and \( X_{TB} \).

(Compare page 241) Brief discussion of infinite state spaces with no boundary.

Example: Coin tossing walk, \( p \uparrow q \downarrow \) on \( \mathbb{Z}, N_0 := \sum_{n=0}^{\infty} 1(X_n = 0) \). Compute:

\[
\mathbb{E}_0 N_0 = \mathbb{E}_0 \sum_{n=0}^{\infty} 1(X_n = 0) \quad (= \text{expected number of returns to } 0)
\]

\[
= \sum_{n=0}^{\infty} P^n(0,0)
\]

\[
= \sum_{m=0}^{\infty} P^{2m}(0,0)
\]

\[
= \sum_{m=0}^{\infty} \binom{2m}{m} p^m q^m
\]

\[
= \sum_{m=0}^{\infty} \binom{2m}{m} (1/2)^m (4pq)^m
\]
Case $p = q = q/2$: recall Stirling’s formula $n! \sim (n/e)^n \sqrt{2\pi n}$. Then

$$\left(\frac{2m}{m}\right) \left(\frac{1}{2}\right)^{2m} \approx \frac{c}{\sqrt{m}}$$

$$4pq = 1$$

Then $\sum_{n=0}^{\infty} \frac{c}{\sqrt{m}} = \infty$.

Case $p \neq q$:

$$\sum_{n=0}^{\infty} \left(\frac{2m}{m}\right) \left(\frac{1}{2}\right)^{2m} \frac{(4pq)^m}{1 - 4pq} < \infty$$

by comparison with a geometric series with ratio $4pq < 1$ for $p \neq q$.

Thus we have $E_0N_0 = \infty$ for $p = q = 1/2$ (recurrent case) and $E_0N_0 < \infty$ for $p \neq q$ (transient case). In the transient case, $E_0N_0 < \infty$ implies $P_0(N_0 < \infty) = 1$, meaning that the random walk visits state 0 only a finite number of times. The same argument shows that in the transient case, for each fixed state $j$, $E_0N_j < \infty$ and hence $P_0(N_j < \infty) = 1$, meaning that the random walk visits state $j$ only finite number of times with probability one. It follows that in the transient case, with probability one the random walk eventually leaves each finite set of states, and hence that with probability one $|X_n| \to \infty$ as $n \to \infty$.

In the recurrent case, the reason why $E_0N_0 = \infty$ is that $P_0(N_0 = \infty) = 1$. This is not fully obvious yet, but argued in next lecture. So in the recurrent case, the random walk keeps coming back to its initial state, infinitely often, with probability one.