Previous lecture showed the value of identifying a sequence with a corresponding generating function defined by a power series, then recognizing that various operations on sequences correspond to familiar algebraic and analytic operations on the generating functions. For instance, if sequences \((f_n)\) and \((g_n)\) have generating functions

\[
F(z) := \sum_{n=0}^{\infty} f_n z^n \quad \text{and} \quad G(z) := \sum_{n=0}^{\infty} g_n z^n
\]

then the convolution

\[
h_n = \sum_{m=0}^{n} f_m g_{n-m}
\]

has generating function

\[
H(z) := \sum_{n=0}^{\infty} h_n z^n = F(z)G(z)
\]

- Discrete random variable \(X\) with values 0, 1, 2, 3,\ldots
  Probability \(\mathbb{P}(X = n) = p_n, n = 0, 1, 2,\ldots\)
  Probability GF (of \(X\), or of \(p_0, p_1,\ldots\))

\[
\phi(s) := \phi_X(s) := \sum_{n=0}^{\infty} p_n s^n, \quad |s| \leq 1
\]

The generic notation is \(\phi(s)\). The subscript \(X\) is just used to indicate what random variable \(X\) the generating function is derived from.
• Basic Properties:

$$\phi(1) = 1 \quad [\text{Assuming } \mathbb{P}(0 \leq X < \infty) = 1]$$

$$\phi(0) = p_0 = \mathbb{P}(X = 0)$$

$$\phi'(s) = \sum_{n=0}^{\infty} np_n s^{n-1}, \quad |s| < 1$$

$$\phi'(0) = p_1$$

$$\phi'(1) = \sum_{n=0}^{\infty} np_n = \mathbb{E}X$$

$$\phi''(s) = \sum_{n=0}^{\infty} n(n-1)p_n s^{n-1}, \quad |s| < 1$$

$$\phi''(0) = 2p_2$$

$$\phi''(1) = \sum_{n=0}^{\infty} n(n-1)p_n = \mathbb{E}(X(X-1))$$

and so on for higher derivatives: evaluating the \(k\)th derivative at 0 gives \(k!p_k\), and evaluating at 1 gives \(E[X(X-1)\cdots(X-k+1)]\). In case any of these moments are infinite, so is the derivative evaluated as a limit as \(s \uparrow 1\). In particular, the first two derivatives of \(\phi\) at 1 give the mean and variance: So

$$E(X) = 1\phi'(1)$$

$$E(X^2) = 1\phi''(1) + \phi'(1)$$

$$Var(X) = E(X^2) - [E(X)]^2 = \phi''(1) + \phi'(1) - (\phi'(1))^2$$

• Uniqueness:

For \(X\) with values \(\{0, 1, 2, \ldots\}\), the function \(s \mapsto \phi_X(s)\) for \(|s| \leq 1\), or even for \(|s| < \epsilon\) for any \(\epsilon > 0\), determines the distribution of \(X\) uniquely.

Proof: \(\mathbb{P}(X = n) = \phi_X^{(n)}(0)/n!\).

• Sums of independent r.v.'s

Write \(\phi_X(s)\) for the GF of \(X\), \(\phi_Y(s)\) for the GF of \(Y\). Assume \(X\) and \(Y\) are independent. Then

$$\phi_{X+Y}(s) = \phi_X(s)\phi_Y(s)$$
because
\[
\phi_X(s)\phi_Y(s) = \left( \sum_{k=0}^{\infty} \mathbb{P}(X = k)s^k \right) \left( \sum_{m=0}^{\infty} \mathbb{P}(Y = m)s^m \right)
= \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} \mathbb{P}(X = k)\mathbb{P}(Y = n - k) \right) s^n
= \sum_{n=0}^{\infty} \mathbb{P}(X + Y = n)s^n
\]

Alternative proof:

\[
\phi_X(s) = \sum_{n=0}^{\infty} \mathbb{P}(X = n)s^n = \mathbb{E}[s^X]
\]

\[
\phi_{X+Y}(s) = \mathbb{E}[s^{X+Y}] = \mathbb{E}[s^X \cdot s^Y] = \mathbb{E}[s^X]\mathbb{E}[s^Y]
\]
because independence of \(X\) and \(Y\) implies independence of \(s^X\) and \(s^Y\), and using the rule for expectation of a product of independent random variables.

- Exercise: Use probability GF to confirm that the sum of independent Poisson’s is Poisson.
  1) Compute GF of \(\text{Poi}(\lambda)\), where \(p_n = e^{-\lambda}\lambda^n/n!:\)
  \[
  \phi(s) = \sum_{n=0}^{\infty} e^{-\lambda}\frac{\lambda^n}{n!} s^n = e^{\lambda s - \lambda} = e^{\lambda(s-1)}.
  \]
  2) Look at product of \(\text{Poi}(\lambda)\) and \(\text{Poi}(\mu)\)’s GF:
  \[
  e^{\lambda(s-1)}e^{\mu(s-1)} = e^{(\lambda+\mu)(s-1)}
  \]
is the GF of \(\text{Poi}(\lambda + \mu)\). That is,
\[
X \sim \text{Poi}(\lambda), \; Y \sim \text{Poi}(\mu), \; X \perp Y \Rightarrow X + Y \sim \text{Poi}(\lambda + \mu)
\]

- Random sums
Suppose \(X_1, X_2, \ldots\) are i.i.d. on \(\{0, 1, 2, \ldots\}\). \(N\) is a random index independent of \(X_1, X_2, \ldots\).

Problem: Find the distribution of \(X_1 + \cdots + X_N = S_N\). (Note: \(S_0 = 0\) by convention)
Solution: Apply GF. Let $\phi_X(s)$ be the GF of the $X_i$'s.

$$
\phi_X(s) = E[s^X] = \sum_{n=0}^{\infty} P(X = n) s^n, \quad X = X_i, \text{for any } i
$$

Compute by conditioning on $N$:

$$
\phi_X(s) = E[s^X] = \sum_{n=0}^{\infty} P(N = n) E[s^{S_N}|N = n]
$$

$$
= \sum_{n=0}^{\infty} P(N = n) E[s^{X_1+\cdots+X_n}]
$$

$$
= \sum_{n=0}^{\infty} P(N = n) [\phi_X(s)]^n
$$

$$
= \phi_N(\phi_X(s))
$$

From the GF of $S_N, \phi_N(\phi_X(s))$, we get formulas for means and variances. Compare with text, first chapter.

**Example: Poisson Thinning** This is a Stat 134 exercise, much simplified by use of GF. Let $X_1, X_2, \ldots$ be independent 0/1 Bernoulli(p) trials. Let $N$ be $\text{Poi}(\lambda)$, independent of the $X_i$'s.

Let $S_N = X_1 + \cdots + X_N = \# \text{ of successes in } \text{Poi}(\lambda) \# \text{ of trials}.$

Then $S_N \sim \text{Poi}(\lambda p)$. This can be checked directly, most easily by showing also that $S_N$ and $N - S_N$ are independent and $S_N \sim \text{Poi}(\lambda(1-p))$. But the GF computation is very quick:

$$
\phi_{S_N}(s) = \phi_N(\phi_X(s))
$$

$$
= \phi_N(q + ps)
$$

$$
= e^{\lambda(q + ps - 1)}
$$

$$
= e^{\lambda p(s-1)}
$$

This is the GF of $\text{Poi}(\lambda p)$, hence the conclusion, by uniqueness of the GF.

**Compare: Moment GF**

Usually the Moment GF is defined for a real valued $X$ as

$$
M_X(t) := E[e^{tX}] = \sum_{n=0}^{\infty} \frac{t^n}{n!} E[X^n]
$$

provided the series converges in some neighbourhood of $t = 0$. For discrete $X$ with values in $\{0, 1, 2, \ldots\}$ the change of variables: $e^t = s$ shows that

$$
M_X(t) = \phi_X(e^t) \text{ and } \phi_X(s) = M_X(\log s)
$$
• **Application of GF to recover a discrete distribution from its factorial moments.**

Consider a discrete distribution of $X$ on $\{0, 1, 2, \ldots, n\}$. Let us check that such a distribution is determined by its binomial moments:

$$
\mathbb{E}
\left(
\frac{X^k}{k!}
\right) = \frac{\mathbb{E}[\binom{X}{k}]}{k!}
$$

where $(X)_k := X(X-1) \cdots (X-k+1)$ is a falling factorial function of $X$, and $\mathbb{E}[\binom{X}{k}]$ is the $k$th factorial moment of $X$. Note that the $k$th binomial moment is just some linear combination of the first $k$ moments of $X$. Now

$$
\phi_X(s) = \mathbb{E}[s^X] = \phi_X(1 + (s - 1)) = \sum_{k=0}^{n} \phi_X^{(k)}(1) \frac{(s - 1)^k}{k!} = \sum_{k=0}^{n} \mathbb{E}\left(\binom{X}{k}\right) (s - 1)^k
$$

Hence for $0 \leq j \leq n$

$$
P(X = j) = \text{Coefficient of } s^j \text{ in } \phi_X(s) = \sum_{k=j}^{n} (-1)^{k-j} \binom{k}{j} \mathbb{E}\left(\binom{X}{k}\right)
$$

In particular, if $X$ is the number of events that occur in some sequence of events $A_1, \ldots, A_n$, in terms of indicators $X_i = 1_{A_i}$

$$
X := \sum_{i=1}^{n} X_i
$$

and then

$$
\binom{X}{k} = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} \prod_{j=1}^{k} X_{i_j}
$$

is the number of ways to choose $k$ distinct events $A_i$ among those which happen to occur. So

$$
\mathbb{E}\left(\binom{X}{k}\right) = \sum_{1 \leq i_1 < i_2 < \cdots < i_k} \mathbb{P}(A_{i_1}A_{i_2} \cdots A_{i_k})
$$
is the usual sum appearing in the inclusion exclusion formula for the probability of a union of \( n \) events, which can be written in present notation as

\[
P(\bigcup_{i=1}^{n} A_i) = P(X \geq 1) = \sum_{k=1}^{n} (-1)^{k-1} \mathbb{E}\left(\frac{X}{k}\right)
\]

Since \( P(X \geq 1) = 1 - P(X = 0) \) and \( \binom{X}{0} = 1 \), this agrees with the previous formula for \( P(X = j) \) in the case \( j = 0 \), which is

\[
P(X = 0) = \sum_{k=0}^{n} (-1)^k \mathbb{E}\left(\frac{X}{k}\right) = 1 + \sum_{k=1}^{n} (-1)^k \mathbb{E}\left(\frac{X}{k}\right)
\]

- **Application to the matching problem**

Let \( M_n \) be the number of matches, that is \( i \) with \( i = \pi_n(i) \), where \( \pi_n \) is a uniformly distributed random permutation of \( \{1, \ldots, n\} \). Apply the previous discussion with \( X = M_n \) and \( A_i \) the event \( i = \pi_n(i) \) to see that

\[
\mathbb{E}\left(M_n\right) = \sum_{1 \leq i_1 < i_2 < \cdots < i_k} \mathbb{P}(A_{i_1} A_{i_2} \cdots A_{i_k}) = \binom{n}{k} \frac{1}{(n)_k} = \frac{1}{k!}
\]

and hence

\[
\mathbb{P}(M_n = j) = \sum_{k=j}^{n} (-1)^{k-j} \binom{k}{j} \frac{1}{k!} = \frac{1}{j!} \sum_{k=j}^{n} (-1)^{k-j} \frac{1}{(k-j)!}
\]

which converges as \( n \to \infty \) to the limit

\[
\frac{1}{j!} \sum_{k=j}^{\infty} \frac{(-1)^{k-j}}{(k-j)!} = \frac{e^{-1}}{j!} = P(M_{\infty} = j)
\]

for a random variable \( M_{\infty} \) with Poisson(1) distribution. Thus the limit distribution of the number of matches in a large random permutation is Poisson(1).