Lecture 17 : Long run behaviour of Markov chains

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- Basic Case: $S$ is finite
- Markov matrix is $P$
- Assume that for some power of $P$ has all entries $> 0$: $\exists k$ such that $P^k(i, j) > 0 \forall i, j \in S$
- Such $P$ is called regular
- Then (Theorem): $\exists$ a unique stationary probability distribution $\pi$ such that:
  - $\pi P = \pi$, meaning $\sum_i \pi_i P(i, j) \forall j \in S$
  - This $\pi$ is then the limit distribution of $X_n$ as $n \to \infty$, no matter what the initial distribution of $X_0$ i.e.
  - $\lim_{n \to \infty} P^n(i, j) = \lim_{n \to \infty} P_i(X_n = j) = \pi_j$ for all $j \in S$.
- Idea of most proofs:
  - Show that $\pi$ exists
  - Show $\lim_{n \to \infty} P^n(i, j) = \pi_j$
  - Uniqueness of $\pi$ is easy.
- Explicit representations of $\pi$
  - My favorite (not in text): let $T_i = \inf\{n : n > 1, X_n = i\}$ where $\inf \phi = \infty$.
  - Then $E_i(\#$ of visits to $j$ before $T_i) = \frac{\pi_j}{\pi_i}$ where $\pi$ is the unique invariant probability distribution
  - Notice: by $\sum_j$ we get $E_i(T_i) = \frac{1}{\pi_i}$, hence
  - $\pi_i = \frac{1}{E_i(T_i)}$
This brings us to discussion of how to solve $\pi P = \pi$. In general, for complicated $P$, this can be a pain. But a simplifying method is often available: look for a reversible equilibrium. This may not exist, but if it does, it’s easy to compute.

### Reversible Equilibrium

- Recall: $\pi = P\pi$ means for the Markov chain $(X_0, X_1, \ldots)$ with transition matrix $P$ that
- $X_0 \sim \pi \implies X_1 \sim \pi$
- That is $X_0 \xrightarrow{d} X_1$ (equality in distribution)
- Ordinary Equilibrium: $\implies X_n \xrightarrow{d} X_0$
- But sometimes, we also have Reversible Equilibrium
- $(x_0, x_1) \xrightarrow{d} (x_1, x_0)$ (equality of joint distributions)
- Look at the probability that either side has value $(i, j)$ to derive the equations
- $\pi_i P(i, j) = \pi_j P(j, i)$ for all pairs of states $i, j$
- Note: if $|S| = N$, we used to have $N$ equations and $N$ unknowns.
- Now $\binom{N}{2} = \frac{N(N-1)}{2}$ equations, $N$ unknowns
- Note that reversibility of 2 steps implies reversibility of $n$ steps: $(x_0, x_1, \ldots, x_n) \xrightarrow{d} (x_n, \ldots, x_1, x_0)$

### Example 1:

- Any random walk on an interval of integers which has only transitions of $+1, -1, 0$ can be solved with reversible equilibrium.
- e.g. on states $\{0, 1, 2\}$

$$P = \begin{bmatrix}
\frac{1}{3} & \frac{1}{2} & 0 \\
\frac{1}{3} & 0 & \frac{2}{3} \\
0 & 1 & 0
\end{bmatrix}$$

- Look for a reversible equilibrium:
- Start with generic $\pi_0$
- $\pi_0 \frac{1}{2} = \pi_1 \frac{1}{3} \implies \pi_1 = \pi_0 \frac{3}{2}$
- $\pi_1 \frac{2}{3} = \pi_2 1 \implies \pi_2 = \pi_1 \frac{2}{3} = \pi_0$
- $0 = 0$
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- Normalize: \( \pi_0 + \pi_1 + \pi_2 = 1 \)
- \( \pi_0 + \pi_0 \frac{3}{2} + \pi_0 = 1 \)
- \( \pi_0(1 + \frac{3}{2} + 1) = 1 \)
- \( \pi_0 = \frac{2}{7} \)
- \( \pi_1 = \frac{3}{7} \)
- \( \pi_2 = \frac{2}{7} \)

• Example 2:
  - Random walk on states in a circle. Take a circle with 5 points, with the probability of moving clockwise, \( p \), and the probability of moving counter clockwise is \( q \)
  - Easy to check that if \( \pi \) is uniform then \( \pi P = \pi \)
  - Reversible only if \( p = q = 1/2 \).

• Example 3:
  - Random walk on \( \{0, 1, 2, \ldots\} \)
  - reflection at 0: \( P(0, 1) = 1 \)
  - \( P(i, i + 1) = p \) for \( i \geq 1 \)
  - \( P(i, i - 1) = q \) for \( i \geq 1 \)
  - Cases:
    * Transient if \( p > q \)
    * Null-recurrent if \( p = q \)
    * Positive Recurrent (similar behavior to regular) if \( p < q \)
  - \( \pi_i = \frac{1}{E_i(\pi_i)} \)
  - For the case: \( p < q \), find there is a unique solution of \( \pi P = \pi \), \( \sum_j \pi_j = 1 \).
  - Find it as a reversible equilibrium:
    * \( \pi_0 1 = \pi_1 q \implies \pi_1 = \frac{\pi_0 q}{q} \)
    * \( \pi_1 p = \pi_2 q \implies \pi_2 = \frac{\pi_0 p}{q} \)
    * \( \pi_n = \pi_0 \frac{1}{q} \left( \frac{p}{q} \right)^{n-1} \)
    * \( 1 = \sum_{n=0}^{\infty} \pi_n = \pi_0 + \frac{\pi_0}{q} + \frac{\pi_0}{q} \left( \frac{p}{q} \right) + \frac{\pi_0}{q} \left( \frac{p}{q} \right)^2 + \ldots = \pi_0 (1 + \frac{1}{q} - \frac{1}{1 - p/q}) \)