• Poisson processes

General idea: Want to model a random scatter of points in some space.

− # of points is random
− locations are random
− want model to be simple, flexible and serve as a building block

Variety of interpretations:

− Natural phenomena such as weather, earthquakes, meteorite impacts
− points represent discrete “events”, “occurrences”, “arrivals” in some continuous space of possibilities
− Continuous aspect: location/attributes of points in physical space/time
− Discrete aspect: counts of numbers of points in various regions of space/time give values in $\rightarrow \{0, 1, 2, 3, \ldots\}$

Abstractly: A general, abstract space $S$. Suitable subsets $B$ of $S$ for which we discuss counts, e.g. intervals, regions for which area can be defined, volumes in space, ...

Generically $N(B)$ should be interpreted as the number of occurrence/arrivals/events with attributes in $B$. Then by definition, if $B_1, B_2, \ldots, B_n$ are disjoint subsets of $S$, then

$$N(B_1 \cup B_2 \cup \cdots \cup B_n) = N(B_1) + \cdots + N(B_n)$$

Simple model assumptions that lead to Poisson processes:

(1) Independence: If $B_1, B_2, \ldots, B_n$ are disjoint subsets of $S$, then the counts $N(B_i)$ are independent.

(2) No multiple occurrences. Example: we model #’s of accidents, not #’s of people killed in accidents, or numbers of vehicles involved in accidents.
It can be shown in great generality that these assumptions imply the Poisson model: \( N(B) \sim \text{Poisson}(\mu(B)) \) for some measure \( \mu \) on subsets \( B \) of \( S \). If \( \mu \) is \( \lambda \) times some notion of length/area/volume on \( S \), then the process is called \textit{homogeneous} and \( \lambda \) is called the rate. In general, \( \mu(B) \) is the expected number of points in \( B \). In the homogeneous case, the rate \( \lambda \) is the expected number of points per unit of length/area/volume.

**Specific**

Most basic study is for a Poisson arrival process on a time line \([0, \infty)\) with arrival rate \( \lambda \).

\[
\begin{array}{cccc}
  \times & \times & \times & \times \\
  a & b & & t \\
\end{array}
\]

\((a, b] = \text{interval of time}\)

\((a, b] \cup (b, c] = (a, c] \)

\[
N(a, b] := \text{# of arrivals in } (a, b] \sim \text{Poisson} \left( \lambda \overbrace{(b-a)}^{\text{Length}} \right)
\]

\(N_t := N(0, t]. \ (N_t, t \geq 0) \) continuous time counting process:

- only jumps by 1
- stationary independent increments
- for \( 0 \leq s \leq t \), \( N_t - N_s \overset{d}{=} N_{t-s} \overset{d}{=} \text{Poisson}(\lambda(t-s)) \)

Define

\[
T_r := W_1 + W_2 + \cdots + W_r = \inf \{ t : N_t = r \}, \quad 0 = T_0 < T_1 < T_2 \cdots \\
W_i := T_{i+1} - T_i
\]
• **Fact/Theorem**

\((N_t, t \leq 0)\) is a Poisson process with rate \(\lambda \iff W_1, W_2, W_3, \ldots\) is a sequence of iid \(\text{Exp}(\lambda)\).

\[
(W_1 > t) \iff (N_t = 0)
\]

\[
P(W_1 > t) = P(N_t = 0) = e^{-\lambda t}
\]

• **Fundamental constructions: Marked Poisson point process**

Assume space of marks is \([0,1]\) at first.

- Points arrive according to a \(\text{PP}(\lambda)\)
- Each point is assigned an independent uniform mark in \([0,1]\)

Let the space of marks be split into two subsets of length \(p\) and \(q\) with \(p+q = 1\).

Each point is a \(\boxtimes\) with probability \(p\) and \(\otimes\) with probability \(q\). Let

\[
\tilde{N}_\boxtimes(t) : \# \text{ of } \boxtimes \text{ up to } t
\]

\[
N_\otimes(t) : \# \text{ of } \otimes \text{ up to } t
\]
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Then

\[ N_{\succeq}(t) \sim \text{Poisson}(\lambda pt) \]
\[ N_{\ominus}(t) \sim \text{Poisson}(\lambda qt) \]
\[ N(t) = N_{\succeq}(t) + N_{\ominus}(t) \]

These relations hold because of the \textit{Poisson thinning property}: if you have Poisson number of individuals with mean \( \mu \), then

- keep each individual with probability \( p \)
- discard each individual with probability \( q \)

Then \# kept and \# discarded are independent Poissons with means \( \mu p \) and \( \mu q \), respectively. Pushing this further, \((N_{\succeq}(t), t \geq 0)\) and \((N_{\ominus}(t), t \geq 0)\) are two independent homogeneous Poisson processes with rates \( \lambda p \) and \( \lambda q \) respectively.

Take a fixed region \((t, u)\) space, count the number of \((T_i, U_i)\) with \((T_i, U_i) \in R\). In the graph below, there are two. Since the sum of independent Poissons is Poisson, the number of \((T_i, U_i)\) has to be Poisson as well.

- Here we simply suppose the marks are uniform
- Generalize to marks which are independent identical with density \( f(x) \)
- \( T_1, T_2, \ldots \) are time arrivals of PP(\( \lambda \))
- \( X_1, X_2, \ldots \) are independent identical with \( \mathbb{P}(X_i \in dx) = f(x)dx \).
- Then \((T_1, X_1), (T_2, X_2), \ldots \) are the points of a Poisson process in \([0, \infty) \times \mathbb{R}\) with measure density \( \lambda dt f(x)dx \)
Here, $N_B \sim \text{Poisson}$ with mean $\int_B \lambda \, dt \int f(x) \, dx$.

- **Practical example** (Queueing model)
  - Customers arrive according to a PP$(\lambda)$
  - A customer arrives at time $t$ and stays in the system for a random time interval with density $f(x)$.

Find a formula for the distribution of \# of customers in system at time $t$.

Mark each arrival by time spent in the system.

$$\mu = \int_0^t \lambda \, ds \int_{t-s}^\infty f(x) \, dx$$

$$= \int_0^t \lambda \, ds(1 - F(s)), \quad \text{where } F(s) = \int_0^s f(x) \, dx$$

Actual number is Poisson with mean $\mu$ calculated above.

- **Connection between PP and uniform order statistics**

**Construction of a PP with rate $\lambda$ on [0,1]**

Step 1: Let $N_1 \sim \text{Poisson}(\lambda)$. This will be the total number of points in [0,1]
Step 2: Given $N_1 = n$, let $U_1, \ldots, U_n$ be independent and uniform on $[0,1]$. Let the points of PP be at $U_1, U_2, \ldots, U_n$. Let

$$T_1 = \min_{1 \leq i \leq n} U_i$$

$$T_k = \min \{ U_i : U_i > T_{k-1}, 1 \leq i \leq n \} \text{ for } k > 1$$

We can only define $T_1, T_2, \ldots, T_n$ ordered values of $U_1, U_2, \ldots, U_n$. Pick all the $T_r$'s with $T_r \leq 1$ this way.

Create $(N_t, 0 \leq t \leq 1) : N_t = \# \{ i : U_i \leq t \} = \sum_{i=1}^{\infty} 1(N_1 \geq i, U_i \leq t)$

Claim: $(N_t, 0 \leq t \leq 1)$ is the usual PP($\lambda$)

This means

- $N_t \sim$ Poisson($\lambda t$)
- $N_{t_1}, N_{t_2} - N_{t_1}, \ldots, N_{t_n} - N_{t_{n-1}}, N_1 - N_{t_n}$ are independent for $0 \leq t_1 \leq \cdots \leq t_n \leq 1$
- $N_t - N_s \sim$ Poisson($\lambda(t-s)$)

Why?

$N_t \sim$ Poisson($\lambda t$) by thinning.

Independence also by thinning: I can assign Poisson($N_1$) counts into categories $(0, t_1], (t_1, t_2], \ldots, (t_n, 1]$ with probabilities $t_1, t_2 - t_1, \ldots, 1 - t_n$.

Poissonization of multinomials:

$$\mathbb{P}(N_{t_1} = n_1, N_{t_2} - N_{t_1} = n_2, \ldots, N_1 - N_{t_{k-1}} = n_k | n_1 + n_2 + \cdots + n_k = n)$$

$$= \mathbb{P}(N_1 = n_1 + \cdots + n_k) \mathbb{P}(N_{t_1} = n_1, N_{t_2} - N_{t_1} = n_2, \ldots, N_1 - N_{t_{k-1}} = n_k | N_1 = n_1 + \cdots + n_k)$$

$$= e^{-\lambda} \frac{\lambda^{n_1+\cdots+n_k}}{(n_1+\cdots+n_k)!} \left( \frac{n_1 + \cdots + n_k}{n_1, n_2, \ldots, n_k} \right) t_1^{n_1} (t_2 - t_1)^{n_2} \cdots (1 - t_{k-1})^{n_k}$$

$$= e^{-\lambda} \frac{(\lambda t_1)^{n_1}}{n_1!} e^{-\lambda(t_2-t_1)} \left( \frac{(\lambda(t_2 - t_1))^{n_2}}{n_2!} \right) \cdots e^{-\lambda(1-t_{k-1})} \left( \frac{(\lambda(1 - t_{k-1}))^{n_k}}{n_k!} \right)$$

Therefore $N_{t_1}, N_{t_2} - N_{t_1}, \ldots, N_{t_n} - N_{t_{n-1}}, N_1 - N_{t_n}$ are independent Poisson as claimed.