• **Queueing Models**

The simplest setup is a single server with some service rules takes a random to process then the customer departs.

\[
\begin{array}{c}
\circ \circ \circ \circ \circ \\
\uparrow \\
\circ \circ \circ \circ \\
\downarrow \\
\text{Queue} \\
\circ \circ \circ
\end{array}
\]

Generically, \( X(t) = \# \) of customers in system(queue) at time \( t \), and model for evolution of \( X \).

• **M/M/1**

  - \( M = \) Markov
  - First \( M = \) for input stream
  - Second \( M = \) for service process
  - \( 1 = \# \) of servers
  - Poisson arrival at rate \( \lambda \)
  - Independent exponential service time with rate \( \mu \)
  - Each customer that is served takes \( \exp(\mu) \) time
  - The service times \( T_1, T_2, \ldots \) are i.i.d. \( \exp(\mu) \) random variables independent of the arrival stream
$q_{ij} = \text{rate of transition from } i \text{ to } j$

$= \text{off diagonal elements of the } A \text{ matrix}$

$q_0 = \lambda = -A_{00}$

$q_{01} = \lambda$

$q_{0j} = 0, \quad \text{for all } j \neq 0, j \neq 1$

$q_1 = \lambda + \mu = A_{11}$

$q_{10} = \mu$

$q_{12} = \lambda$

$q_{1j} = 0, \quad \text{for all } j \neq 0, j \neq 1, j \neq 2$

• Condition for stability of the queue: $\lambda < \mu$

$\lambda = \mu$ is exceptional (null-recurrent chain).

$\lambda > \mu$ is linear growth (like a RW with upward drift: transient chain).

Stability means \( \lim_{t \to \infty} P(X_t = j) = \pi_j > 0 \) with \( \sum_j \pi_j = 1 \).

We can calculate \( \pi \). How?

\[
\pi P_t = \pi, \quad \text{for all } t
\]

Take \( \frac{d}{dt} \) and evaluate at 0 \( \implies \pi A = 0 \). (Note: we did not try to solve differential equations for \( P_t \). In the M/M/1 queue \( P_t(i, j) \) is a nasty power series (involving Bessel functions))

Use \( q_i = -A_{ii}, \ q_{ij} = A_{ij}, i \neq j, \) and \( \sum_i \pi_i A_{ij} = 0, \) for all \( j \)

We get

\[
\forall j, \quad \sum_{i \neq j} \mu_i q_{ij} - \pi_j q_j = 0
\]

\( \implies \forall j, \quad \sum_{i \neq j} \mu_i q_{ij} = \pi_j q_j \)
That is, at equilibrium,

Rate in to $j = \text{Rate out of } j$

**Fact:** For birth/death chain on integers, the only possible equilibrium is reversible. So always look for solution of Rev EQ.

\[
\begin{bmatrix}
\pi_0 \lambda &= \pi_1 \mu \\
\pi_1 \lambda &= \pi_2 \mu \\
\vdots
\end{bmatrix}
\Rightarrow \frac{\pi_n}{\pi_{n-1}} = \frac{\lambda}{\mu} \Rightarrow \pi_n = \pi_0 (\frac{\lambda}{\mu})^n
\]

Use $\sum_{n=0}^{\infty} \pi_n = 1$, get

\[
\pi_n = (\frac{\lambda}{\mu})^n (1 - \frac{\lambda}{\mu}), \quad n = 0, 1, 2, \ldots
\]

Check P549(2.9)

In the limit, mean queue length $= 1/(1 - \frac{\lambda}{\mu}) = \frac{\mu}{\mu - \lambda}$. If $\lambda$ is close to $\mu$, this is large. Fix $\lambda$, as $\mu \downarrow \lambda$, $\frac{\mu}{\mu - \lambda} \uparrow \infty$.

- **M/M/$\infty$**
  - Input rate $\lambda$
  - Unlimited number of servers, each operate at rate $\mu$ on one customer at a time

Use the fact that mean of $\exp(\mu_1), \exp(\mu_2), \ldots, \exp(\mu_n)$ is $\exp(\mu_1 + \mu_2 + \cdots + \mu_n)$.

Condition for stability? No matter what $\lambda > 0, \mu > 0$, the system is stable. To see this, establish a Rev EQ probability distribution:
\[
\begin{bmatrix}
\pi_0 \lambda &=& \pi_1 \mu \\
\pi_1 \lambda &=& \pi_2 2\mu \\
\pi_2 \lambda &=& \pi_3 3\mu \\
\vdots
\end{bmatrix}
\Rightarrow \frac{\pi_n}{\pi_{n-1}} = \frac{\lambda}{n\mu} \Rightarrow \pi_n = \pi_0 \left(\frac{\lambda}{\mu}\right)^n \frac{1}{n!}
\]

So \(\pi_0 = e^{-\lambda/\mu}\). The equilibrium probability distribution is \(\text{Poisson}(\lambda/\mu)\) and the mean queue length is \(\lambda/\mu\).