• Brownian Motion

Idea: Some kind of continuous limit of random walk.

Consider a random walk with independent $\pm \Delta$ steps for some $\Delta$ with $\frac{1}{2} \uparrow \frac{1}{2} \downarrow$, and $N$ steps per unit time. Select $\Delta = \Delta_N$, so the value of the walk at time 1 has a limit distribution as $N \to \infty$.

Background: Coin tossing walk with independent $\pm 1$ steps with $\frac{1}{2} \uparrow \frac{1}{2} \downarrow$.

$$S_N = X_1 + \cdots + X_N$$
$$\mathbb{E}S_N = 0$$
$$\mathbb{E}S_N^2 = N$$

If instead we add $\pm \Delta_N$, get $S_N \Delta_N$ and

$$\mathbb{E}(S_N \Delta_N)^2 = \mathbb{E}(S_N^2) \Delta_N^2 = N \Delta_N^2 \equiv 1$$

if we take $\Delta_N = 1/\sqrt{N}$.

Value of our process at time 1 with scaling is then

$$S_N/\sqrt{N} \xrightarrow{d} \text{Normal}(0, 1) \quad \text{as } N \to \infty$$

according the the normal approximation to the binomial distribution. The same is true for any distribution of $X_i$ with mean 0 and variance 1, by the Central
Scaled process at time $t$ has value

$$\frac{S_{\lfloor tN \rfloor}}{\sqrt{\frac{|tN|}{N}}} = \frac{S_{\lfloor tN \rfloor}}{\sqrt{[tN]}} \sqrt{\frac{tN}{N}}$$

Fix $0 < t < 1$, $N \to \infty$,

$$\frac{S_{\lfloor tN \rfloor}}{\sqrt{[tN]}} \to N(0, 1) \quad \text{and} \quad \sqrt{\frac{|tN|}{N}} \to \sqrt{t}$$

So

$$\frac{S_{\lfloor tN \rfloor}}{\sqrt{N}} \xrightarrow{d} \text{Normal}(0, t)$$

the normal distribution with mean 0, variance $t$, and standard deviation $\sqrt{t}$. In the distributional limit as $N \to \infty$, we pick up a process $(B_t, t \geq 0)$ with some nice properties:

- $B_t \sim \text{Normal}(0, t)$, $\mathbb{E}B_t = 0$, $\mathbb{E}B_t^2 = t$.
- For $0 < s < t$, $B_s$ and $B_t - B_s$ are independent.

$$B_t = B_s + (B_t - B_s)$$

$$\implies B_t - B_s \sim \text{Normal}(0, t - s)$$

$$(B_t|B_s) \sim \text{Normal}(B_s, t - s)$$

For times $0 < t_1 < t_2 < \cdots < t_n$, $B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}}$ are independent Normal$(0, t_i - t_{i-1})$ random variables for $1 \leq i \leq n, t_0 = 0$. This defines the finite dimensional distributions of a stochastic process $(B_t, t \geq 0)$ called a **Standard Brownian Motion** or a **Wiener Process**.

**Theorem:** (Norbert Wiener) It is possible to construct such a Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\mathbb{P}$ is a countably additive probability measure, and the path $t \mapsto B_t(w)$ is continuous for all $w \in \Omega$.

More formally, we can take $\Omega = C[0, \infty) = \text{continuous functions from } [0, \infty) \to \mathbb{R}$. Then for a function (also called a **path**) $w = (w(t), t \geq 0) \in \Omega$, the value of
$B_t(w)$ is simply $w(t)$. This is the canonical path space viewpoint. This way $\mathbb{P}$ is a probability measure on the space $\Omega$ of continuous functions. This $\mathbb{P}$ is called Wiener measure. Note. It is customary to use the term Brownian motion only if the paths of $B$ are continuous with probability one.

- **Context:** Brownian motion is an example of a Gaussian process (Gaussian = Normal). For an arbitrary index set $I$, a stochastic process $(X_t, t \in I)$ is called Gaussian if
  - for every selection of $t_1, t_2, \ldots, t_n \in I$, the joint distribution of $X_{t_1}, X_{t_2}, \ldots, X_{t_n}$ is multivariate normal.
  - Here multivariate normal
    $\iff \sum_i a_i X_{t_i}$ has a one-dimensional normal distribution for whatever choice of $a_1, \ldots, a_n$.
    $\iff X_{t_1}, X_{t_2}, \ldots, X_{t_n}$ can be constructed as $n$ (typically different) linear combinations of $n$ independent normal variables.

Easy fact: The FDD’s of a Gaussian process are completely determined by its mean and covariance functions

$$\mu(t) := \mathbb{E}(X_t) \quad \sigma(s, t) = \mathbb{E}(X_s - \mu(s))(X_t - \mu(t))$$

Here the function $\sigma$ is non-negative definite meaning it satisfies the system of inequalities implied by $\text{Var}(\sum_i a_i X_{t_i}) \geq 0$ whatever the choice of real numbers $a_i$ and indices $t_i$.

Key consequences: If we have a process $\hat{B}$ with the same mean and covariance function as a BM $B$, then the finite-dimensional distributions of $\hat{B}$ and $B$ are identical. If the paths of $\hat{B}$ are continuous, then $\hat{B}$ a BM.

Example: Start with $(B_t, t \geq 0) = \text{BM}$, fix a number $v \geq 0$, look at $(B_{v+t} - B_v, t \geq 0)$. This is a new BM, independent of $(B_s, 0 \leq s < v)$.

- **Time inversion**

Let $\hat{B}_t := \begin{cases} 
    tB(1/t) & \text{for } t > 0 \\
    0 & \text{for } t = 0 
\end{cases}$. Check that $\hat{B}$ is a BM. Note first that $\hat{B}$ is also a Gaussian process.
(1) Continuity of paths
Away from $t = 0$, this is clear. At $t = 0$, does $tB(1/t) \to 0$ as $t \to 0$? Compute $\text{Var}(tB(1/t)) = t^2\text{Var}(B(1/t)) = t \to 0$. With some more care, it is possible to establish path continuity at 0 (convergence with probability one).

(2) Mean and covariances

$$\mathbb{E}(tB(1/t)) = t\mathbb{E}(B(1/t)) = 0 = \mathbb{E}(B_t)$$

Covariances: for $0 < s < t$,

$$\mathbb{E}(B_sB_t) = \mathbb{E}(B_s(B_s + B_t - B_s))$$
$$= \mathbb{E}(B_s^2) + \mathbb{E}(B_s(B_t - B_s))$$
$$= s + 0$$
$$= s$$

$$\mathbb{E}(\hat{B}_s\hat{B}_t) = \mathbb{E}(sB(1/s)tB(1/t))$$
$$= st\mathbb{E}(B(1/s)B(1/t))$$
$$= st \cdot \frac{1}{t} \quad \text{since } \frac{1}{t} < \frac{1}{s}$$
$$= s$$

• Example:
Distribution of maximum on $[0,t]$, $M_t := \sup_{0 \leq s \leq t} B_s$

Fact: $M_t \overset{d}{=} |B_t|$}

$$\mathbb{P}(M_t) = 2\mathbb{P}(B_t > x)$$

$$\mathbb{P}(M_t \in dx) = \frac{2}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx$$
Sketch a proof: We have $B \overset{d}{=} -B$.

\[ \mathbb{P}(M_t > x, B_t < x) = \mathbb{P}(M_t > x, B_t > x) = \mathbb{P}(B_t > x) \]

by a reflection argument. This uses $B \overset{d}{=} -B$ applied to $(B(T_x + v) - x, v \geq 0)$ instead of $B$ (strong Markov property) where $T_x$ is the first hitting time of $x$ by $B$. Add and use $\mathbb{P}(B_t = x) = 0$, to get $\mathbb{P}(M_t > x) = 2\mathbb{P}(B_t > x)$. 