1 Further Analysis of Markov Chains

• Q: Suppose $\lambda$ = row vector, $f$ = column vector, and $P$ is a probability transition matrix, then what is the meaning of $\lambda P^n f$? Here $\lambda P^n f$ is the common value of $(\lambda P^n)f = \lambda(P^n f)$ for usual operations involving vectors and matrices. The above equality is elementary for finite state space. For countable state space it should be assumed e.g. that both $\lambda$ and $f$ are non-negative. Assume that $\lambda$ is a probability distribution, i.e. $\lambda_i > 0$ and $\sum_i \lambda_i = 1$. Suppose then that $X_0, X_1, \ldots, X_n$ is a Markov chain and the distribution of $X_0$ is $\lambda$: $\mathbb{P}(X_0 = i) = \lambda_i$. Then $(\lambda P^n)_j = \mathbb{P}(X_n = j)$. So

$$
\lambda P^n f = \sum_j (\lambda P^n)_j f(j)
$$

$$
= \sum_j \mathbb{P}(X_n = j)f(j)
$$

$$
= \mathbb{E}f(X_n)
$$

So the answer is: $\lambda P^n f = \mathbb{E}f(X_n)$ for a Markov chain with initial distribution $\lambda$ and transition probability matrix $P$.

• Notice that the same conclusion is obtained by first recognizing that $P^n f(i) = \mathbb{E}[f(X_n)|X_0 = i]$:

$$
\lambda P^n f = \lambda(P^n f)
$$

$$
= \sum_i \lambda_i (P^n f)_i
$$

$$
= \sum_i \lambda_i \mathbb{E}[f(X_n)|X_0 = i]
$$

$$
= \mathbb{E}[\mathbb{E}[f(X_n)|X_0]]
$$

$$
= \mathbb{E}f(X_n)
$$
Q: What sort of functions of a Markov chain give rise to Martingales?

Say we have a MC $X_0, X_1, \ldots$ with transition probability matrix $P$. Seek a function $h$ such that $h(X_0), h(X_1), \ldots$ is a Martingale.

Key computation:

$$E[h(X_{n+1})|X_0, X_1, \ldots, X_n] = E[h(X_{n+1})|X_n] = (Ph)(X_n)$$

where

$$Ph(i) = \sum_j P(i,j)h(j)$$

$$= E[h(X_1)|X_0 = i]$$

$$= E[h(X_{n+1})|X_n = i]$$

Notice that if $Ph = h$, then

$$E[h(X_{n+1})|h(X_0), \ldots, h(X_n)] = E[E[h(X_{n+1})|X_0, \ldots, X_n]|h(X_0), \ldots, h(X_n)]$$

$$= E[h(X_n)|h(X_0), \ldots, h(X_n)]$$

$$= h(X_n)$$

**Definition:** A function $h$ such that $h = Ph$ is called a harmonic function associated with the Markov matrix $P$, or a $P$-harmonic function for short.

**Conclusion:** If $h$ is a $P$-harmonic function and $(X_n)$ is a Markov chain with transition matrix $P$, then $(h(X_n))$ is a martingale. You can also check the converse: if $(h(X_n))$ is a martingale, no matter what the initial distribution of $X_0$, then $h$ is $P$-harmonic.

**Note:** A constant function $h(i) \equiv c$ is harmonic for every transition matrix $P$, because every row of $P$ sums to 1.

**Example:** A simple random walk with $\pm 1, (1/2, 1/2)$ on $\{0, \ldots, b\}$, and 0 and $b$ absorbing: $P(0, 0) = 1, P(b, b) = 1, P(i, i \pm 1) = 1/2, \ 0 < i < b$.

Now describe the harmonic function solutions $h = (h_0, h_1, \ldots, h_b)$ of $h = Ph$.

For $0 < i < b$, $h(i) = \sum_j P(i,j)h(j) = 0.5h(i-1) + 0.5h(i+1)$, $h(0) = h(0), h(b) = h(b)$.

Observe that $h = Ph \Rightarrow i \rightarrow h(i)$ is a straight line.

So if $h(0)$ and $h(b)$ are specified, get $h(i) = h(0) + i\frac{h(b) - h(0)}{b}$. So there is just
a two-dimensional subspace of harmonic functions. And a harmonic function is
determined by its boundary values.

**Exercise:** Extend the discussion to a $p \uparrow, q \downarrow$ walk with absorption at 0 and $b$ for
$p \neq q$. (Hint: Consider $(q/p)^i$ as in discussion of the Gambler’s ruin problem
for $p \neq q$)

## 2 First Step Analysis

First step analysis is the general approach to computing probabilities or expecta-
tions of a functional of a Markov chain $X_0, X_1, \ldots$ by conditioning the first
step $X_1$. The key fact underlying first step analysis is that if $X_0, X_1, \ldots$ is a
MC with some initial state $X_0 = i$ and transition matrix $P$, then conditionally
on $X_1 = j$, the sequence $X_1, X_2, X_3, \ldots$ (with indices shifted by 1) is a Markov
chain with $X_1 = j$ and transition matrix $P$. You can prove this by checking
from first principles e.g.

$$P(X_2 = j_2, X_3 = j_3, X_4 = j_4 | X_1 = j) = P(j, j_2)P(j_2, j_3)P(j_3, j_4)$$

- Back to gambler’s ruin problem: Consider random walk $X_0, X_1, \ldots$ with $\pm 1, (1/2, 1/2)$
on $\{0, \ldots, b\}$. We want to calculate $\mathbb{P}$(walk is absorbed at $b$ | starts at $X_0 = a$).

- New approach: let $h_b(i) := \mathbb{P}$(reach $b$ eventually | $X_0 = i$).
  
  Now $h_b(0) = 0, h_b(b) = 1, \text{ and for } 0 < i < b$,
  
  $$h_b(i) = P(i, i + 1)\mathbb{P}$(reach $b$ eventually | $X_1 = i + 1)$
  
  $$+ P(i, i - 1)\mathbb{P}$(reach $b$ eventually | $X_1 = i - 1)$
  
  $$= 0.5 \ h_b(i + 1) + 0.5 \ h_b(i - 1)$$

  Observe that $h = h_b$ solves $h = Ph$. So from the results of harmonic functions,
  $h_b(i) = i/b$.

- Example: Random walk on a grid inside square boundary.
  
  For the symmetric nearest neighbor random walk, with all four steps to nearest
neighbors equally likely, and absorption on the boundary, a function is harmonic
if its value at each state $(i, j)$ in the interior of the grid is the average of its
values at the four neighbors $(i \pm 1, j \pm 1)$. This is a discrete analog of the
notion of a harmonic function in classical analysis, which is a function in a two
dimensional domain whose value at $(x, y)$ is the average of its values around
every circle centered at $(x, y)$ which is inside the domain.