1 Prerequisites

Poisson Random Variables, Poisson Point Processes, Characteristic Functions, \(\infty\)-divisible Laws

2 Summary

The Compound Poisson distribution was introduced in the last topic. In this topic, we formally define the Compound Poisson distribution. We also introduce the Poisson random measure, which is based off of the Poisson point process. Finally, we prove a result about \(\infty\)-divisible laws that leads to the general L-K Formula. See for example section 2.6 of [1].

3 Compound Poisson Distribution

Let \(X_1, X_2, \ldots\) be independent and identically distributed random variables with distribution \(F\) on \(\mathbb{R}\):

\[
F(B) = \mathbb{P}[X_i \in B]
\]

Let \(N_\lambda\) be a Poisson random variable with mean \(\lambda\):

\[
\mathbb{P}[N_\lambda = k] = \frac{\lambda^k e^{-\lambda}}{k!}, \quad k = 0, 1, 2, \ldots
\]

**Theorem 1** Let \(Y_t\) be the sum of a Poisson number, \(N_t\), of i.i.d. random variables with mean \(\lambda t\):

\[
Y_t = \sum_{i=1}^{N_t} X_i.
\]
$N_t$ is the Poisson process with rate $\lambda$ discussed in the previous topic. Then the characteristic function of $Y_t$ is given by:

$$\mathbb{E}[e^{isY_t}] = \exp\left\{\lambda t \int_{\mathbb{R}} (e^{isx} - 1) F(dx)\right\}. $$

Note that this theorem was proved in the previous topic.

**Remark 1:** For any fixed time $t$, we can define a measure $L$ on $\mathbb{R}$, where $L(B) = \lambda t F(B)$ for $B \in \mathcal{B}$. Then in general, the probability distribution associated with the characteristic function

$$\phi(s) = \exp\left\{\int_{\mathbb{R}} (e^{isx} - 1) L(dx)\right\}. $$

is called a Compound Poisson distribution with parameter $L$ (denoted by $\text{CP}(L)$).

**Remark 2:** From the characteristic function of the Compound Poisson, it is clear that

$$\text{CP}(L) \ast \text{CP}(M) = \text{CP}(L + M).$$

Thus, $\text{CP}(L)$ is infinitely divisible since the convolution $n$-th root of $\text{CP}(L)$ is $\text{CP}\left(\frac{L}{n}\right)$.

**Toward Poisson Random Measure:** There is a nice interpretation for the measure $L$. Recall that we constructed the Poisson point process on any $\sigma$-finite measure space. On a space with finite measure $L(S) < \infty$, we get the measure $N$:

$$N(B) = \sum_{i=1}^{N(S)} 1\{X_i \in B\}. $$

where $N(S)$ is a Poisson random variable with mean $L(S)$. To find the distribution of $N(B)$, apply the previous theorem with $X_i$ replaced by $1\{X_i \in B\}$ and set $t = 1$. Recall that

$$F(B) = \mathbb{P}[X_i \in B] = \frac{L(B)}{L(S)}. $$

We get

$$\mathbb{E}[e^{isN(B)}] = \exp\left\{(e^{is} - 1) L(B)\right\}, $$

so $N(B) \sim \text{Poisson}(L(B))$. 

More generally, for $B_1, B_2, ..., B_m$, $m$ disjoint sets, by the same argument,

$$
E\left[e^{i\sum_{k=1}^{m} t_k N(B_k)}\right] = \prod_{k=1}^{m} E\left[e^{i t_k N(B_k)}\right]
$$

and observe that the LHS is the multivariate characteristic function of the vector $(N(B_1), N(B_2), ..., N(B_m))$ at $(t_1, t_2, ..., t_m)$, and the RHS is the multivariate characteristic function of a collection of independent random variables with a Poisson distribution by the above. Consequently, by the uniqueness theorem for multivariate characteristic functions, we conclude that $N(B_1), N(B_2), ..., N(B_m)$ are independent Poisson variables. Recall that this was already proved in a more hands-on way by induction in the previous topic.

## 4 Poisson Random Measure

Start with a measure $L$ defined on a $\sigma$-finite measure space $S$ (most often $\mathbb{R}$). As above, let $N(B) = \sum_{i=1}^{N(S)} 1\{X_i \in B\}$, the point process counting values in $B$ up to $N(S)$ where $N(S)$ was the Poisson random variable with mean $L(S)$. Then a Poisson random measure with intensity $L$ is the collection of random variables $(N(B), B \in \mathcal{B})$. Recall the properties we derived above: If $B_1, ..., B_m$ are disjoint Borel sets, $(N(B_i), 1 \leq i \leq m)$ are independent with distributions Poisson$(L(B_i))$ for $1 \leq i \leq m$, respectively.

**Example 2** Let $0 < T_1 < T_2 < ...$ be a sum of i.i.d. Exponential($\lambda$) random variables. So $N_t = \sum_{i=1}^{\infty} 1\{T_i \leq t\} \sim$ Poisson($\lambda t$). Then $(N_t, 0 \leq t \leq T)$ has the same distribution as $(N[0,t], 0 \leq t \leq T)$ where

$$
N[0,t] = \sum_{i=1}^{N_t} 1\{X_i \leq t\}
$$

for $X_1, X_2, ... \sim U[0,T]$.

This is an example of a famous connection between sums of exponentials and uniform order statistics. It is discussed in exercise 6.7 of [1].

## 5 Link to infinitely-divisible law

**Theorem 3** *(Lévy-Khinchin)* Every $\infty$-divisible distribution on $\mathbb{R}$ is a weak limit of shifted CP distributions.
Look at the characteristic function of a centered CP distribution to discover an interesting connection with Lévy-Khinchin. Take $S \sim CP(L)$ and look at $(S - \mathbb{E}[S])$. Assuming that $\int |x| L(dx) < \infty$,

$$
\mathbb{E}[e^{it(S-\mathbb{E}[S])}] = \exp\{-it\mathbb{E}[S]\} \exp\left\{ \int (e^{itx} - 1) L(dx) \right\} = \exp\left\{ \int (e^{itx} - 1 - itx) L(dx) \right\}
$$

It is easy to check that

$$
\mathbb{E}[(S - \mathbb{E}[S])^2] = \int x^2 L(dx)
$$

From this, we see that in particular this formula defines a characteristic function for every positive measure $L$ on $\mathbb{R}$ with $L(-1,1)^c = 0$ and $\int_{-1}^1 x^2 L(dx) < \infty$.

This observation leads to the general L-K formula, which is the subject of a future topic.

**References**