1 Prerequisites

Uniform Asymptotic Negligibility (UAN), Lévy measures. Infinitely divisible laws

2 Summary

The following finds conditions on \( \{X_{nk}\} \) which are necessary and sufficient for the existence of constants \( b_n \) so that \( S_n - b_n \Rightarrow F \) for some \( F \).

3 The General Central Limit Theorem

Consider a UAN triangular array \( \{X_{nk}\} \) and let \( S_n \) be the \( n \)-th row sum. Assume, as usual, that each row consists of independent random variables.

Let \( L_n(B) = \sum_k P(X_{nk} \in B) \) for \( 0 \notin B \). \( L_n\{0\} = 0 \). Notice that \( L_n \) is a finite measure (bounded by the number of terms in a row) hence \( L_n \) is a Lévy measure.

Define

\[
\tau(x) = x1_{|x|\leq 1} + 1_{x>1} - 1_{x<1},
\]

\[
a_{nk} = E[\tau(X_{nk})], \quad a_n = \sum_k a_{nk}, \quad \text{and}
\]

\[
v_{nk} = Var[\tau(X_{nk})], \quad v_n = \sum_k v_{nk}.
\]

Let \( b_n \) be an arbitrary real sequence.

**Theorem.** \( S_n - b_n \Rightarrow F \) (where \( F \) is a proper distribution function on \( \mathbb{R} \)) if and only if the following three conditions hold:
(i) $L_n \to L$ in the sense that there is some measure $L$ on $\{0\}^c$ such that for each bounded and continuous $f$ which vanishes at the neighborhood of 0 (vanishes in $(-\epsilon, \epsilon)$ for some $\epsilon$) it holds: $\int f dL_n \to \int f dL$.

(ii) $a_n - b_n \to c \in \mathbb{R}$

(iii) $v_n \to v$ for some $v \geq 0$.

**Corollary.** The limit $F$ satisfies

$$\int e^{itx} F(dx) = e^{\Psi_{L, \sigma^2, \epsilon}(t)}$$

where $\sigma^2 = v - \int (\tau(x))^2 L(dx)$.

**Corollary.** Every $\infty$-divisible law has characteristic function of the form $e^{\Psi_{L, \sigma^2, \epsilon}}$. 