1 Prerequisites

Extended distribution function, tightness, weak convergence.

2 Summary

The Helly-Bray selection principle, together with its variation, addresses the question of the conditions under which we can find a subsequence converging to a cumulative distribution function.

3 Helly-Bray Selection Principle

Theorem 1 (Helly-Bray Selection Principle) Every sequence of extended distribution functions $F_n$ has a subsequence $F_{n(k)}$ such that $F_{n(k)} \to F(x)$ for all continuity points $x$ of $F$ for some extended distribution function $F$.

The following lemma is useful in the proof:

Lemma 2 Let $D \subset \mathbb{R}$ be dense. Let $F_n$ be a sequence of e.d.f.s such that $\lim_{n \to \infty} F_n(d) = F_\infty(d)$, $\forall d \in D$. Then, $F_n \Rightarrow F_\star$ where $F_\star(x) := \inf_{x<d \in D} F_\infty(d)$.

The proof of this lemma is left as an exercise to the reader. The following proof is of the Helly-Bray selection principle.

Proof: Let $F_n$ be a sequence of e.d.f.s, and let $D = \{d_1, d_2, \ldots\}$ be any countable, dense set. Because $F_n(d_i)$ takes values in $[0,1]$, it has a converging subsequence $m_1(i), i = \{1, \ldots\}$. Let $F_\infty(d_1)$ be the limit. Similarly, we can find $m_k(i)$, a subsequence of $m_{k-1}(j)$, such that $F_n(q_k)$ converges to $F_\infty(q_k)$ along $m_k$. Using Cantor’s diagonal argument, find a subsequence $n(k) = m_k(k)$ such that $F_{n(k)}(d) \to F_\infty(d)$ for all $d \in D$. Apply the previous lemma and see that $F_{n(k)} \Rightarrow F_\star$. $\blacksquare$
4 Variation of Helly-Bray selection principle

Combining the property of tightness with the Helly-Bray selection principle, we get:

**Theorem 3 (Variation of the Helly-Bray Selection Principle)** Every tight sequence of proper distribution functions $F_n$ has a subsequence $F_{n(k)}$, such that $F_{n(k)} \Rightarrow F$ where $F$ is a proper distribution function.

The proof of this theorem is immediate from the Helly-Bray selection principle and the definition of tightness.