1 Prerequisites

Fubini’s Theorem.

2 Summary

This identity will be used to derive the inversion formula of characteristic function. See Section 2.3 of [1].

3 Parseval’s Identity

Lemma 1 (Parseval’s identity) $X$ and $Y$ are two real random variables with distributions $\mathbb{P}$ and $\mathbb{Q}$.

$$\int \varphi_X(y) \mathbb{Q}(dy) = \int \varphi_Y(x) \mathbb{P}(dx). \quad (1)$$

Proof: Since $f(x, y) = e^{ixy}$ is integrable, we can apply Fubini’s Theorem,

$$\mathbb{E}e^{iXY} = \int \left[ \int e^{i xy} \mathbb{P}(dx) \right] \mathbb{Q}(dy) = \int \varphi_X(y) \mathbb{Q}(dy) = \mathbb{E}\varphi_X(Y)$$

$$= \int \left[ \int e^{i xy} \mathbb{Q}(dy) \right] \mathbb{P}(dx) = \int \varphi_Y(x) \mathbb{P}(dx) = \mathbb{E}\varphi_Y(X)$$

Parseval’s identity (1) is useful: on the left-hand side is the characteristic function $\varphi_X$, and on the right-hand side its probability measure $\mathbb{P}$. The two sides are linked by another random variable $Y$. 
References