1 Prerequisites

Poisson distribution, Geometric distribution, Exponential distribution

2 Summary

Definition and characterization of Poisson processes, with motivation. Definition of a Poisson point process with examples and a proof of existence in the $\sigma$-finite case. See Section 2.6c of [1].

3 Motivation for Poisson Processes

Let $\lambda \geq 0$ and $n \in \mathbb{N} \setminus \{0\}$ and consider the partition of $(0, \infty)$ into the half-open intervals $I_1, I_2, \ldots$ where $I_i = ((i - 1)/n, i/n]$. Take a sequence $X_1, X_2, \cdots$ of independent random variables with $X_i \sim \text{Bernoulli}(\lambda/n)$, i.e., $P(X_i = 1) = \lambda/n$ and $P(X_i = 0) = 1 - \lambda/n$.

We can interpret $X_i = 1$ to mean that a certain event was observed in the time period $I_i$. As $n$ becomes large these time periods become small, and so does the probability of observing an event in any particular one of them, but the expected number of observed events per unit time remains $\lambda$.

For a time $t \geq 0$ of the form $i/n$, the number of events observed in the time period $(0, t] = I_1 \cup \cdots \cup I_i$ is given by $N_t = \#\{j \leq i : X_j = 1\}$. We have $N_t \sim \text{Binomial}(nt, \lambda/n)$, so as $n \to \infty$ the distribution of $N_t$ converges weakly to $\text{Poisson}(\lambda t)$.

Let $T_k$ be the time until the $k$-th event is observed. For any $i,j \in \mathbb{N}$ we have $P(nT_{k+1} - nT_k > j \mid nT_k = i) = (1 - \lambda/n)^j$, so $nT_{k+1} - nT_k \sim \text{Geometric}(\lambda/n)$. Therefore as $n \to \infty$ the distribution of the “waiting time” $T_{k+1} - T_k$ converges weakly to $\text{Exponential}(\lambda)$. 


The random variables $N_t$ above can be considered a discrete-time stochastic process, since they are only defined when $t = i/n$ for $i \in \mathbb{N}$. A Poisson process is the continuous analog of this process, with variables $N_t$ for all real $t \geq 0$ that realize the limiting behavior of the $N_t$ above as $n \to \infty$.

## 4 Definition of a Poisson Process

A (continuous-time) stochastic process is a family $\langle Y_t : t \geq 0 \rangle$ of random variables. We say a stochastic process has stationary independent increments if, given any intervals $(s_1, t_1], \ldots, (s_n, t_n] \subset [0, \infty)$, the “increments” $Y_{t_i} - Y_{s_i}$ are equidistributed when the intervals have equal length and are independent when the intervals are pairwise disjoint.

**Definition 1** A Poisson process with rate $\lambda$ is a stochastic process $\langle N_t : t \geq 0 \rangle$ with stationary independent increments such that $N_t - N_s \sim \text{Poisson}(\lambda(t - s))$ when $0 \leq s \leq t$, and $N_0 = 0$ a.s.

We can construct a Poisson process with rate $\lambda$ by taking independent random variables $X_1, X_2, \ldots \sim \text{Exponential}(\lambda)$, defining $T_k = X_1 + \cdots + X_k$ for $k \in \mathbb{N}$, and defining $N_t = \sup\{n \in \mathbb{N} : T_n \leq t\}$. We can think of $T_k$ as the time until the $k$-th event is observed and $N_t$ as the number of events observed by time $t$. The proof that this is a Poisson process is given in [1, Sec. 2.6c].

The Poisson process constructed this way is a counting process, meaning that at almost every point $\omega$ in the sample space $N_t$ is an increasing, right-continuous, $\mathbb{N}$-valued function of $t$ whose left and right limits at any time $t$ differ by at most 1.

Given any counting process $\langle N_t : t \geq 0 \rangle$ it makes sense to think of the random variable defined by $T_k = \inf\{t \geq 0 : N_t \geq n\}$ as the time at which the $k$-th event is observed, and the random variable $T_{k+1} - T_k$ as the waiting time between the $k$-th and $(k+1)$-st events.

Among counting processes, the Poisson process we have constructed from exponential waiting times is essentially unique:

**Theorem 2** If a counting process $\langle N_t : t \geq 0 \rangle$ is Poisson with rate $\lambda$ then the waiting times $T_{k+1} - T_k$ for $k \in \mathbb{N}$ are independent with distribution $\text{Exponential}(\lambda)$. 

**Proof:** Let \( \langle N_t : t \geq 0 \rangle \) be a Poisson process with rate \( \lambda \). Let \( t_1, \ldots, t_k \in [0, \infty) \) with \( t_1 < \cdots < t_k \). We have

\[
P\left( \bigcap_{j=1}^{k} t_j < T_j \leq t_j + dt_j \right) = P\left( \bigcap_{j=1}^{k} [(N_{t_j} - N_{t_{j-1}+dt_{j-1}} = 0) \cap (N_{t_j+dt_j} - N_{t_j} = 1)] \right)
\]

\[
= \prod_{j=1}^{k} P(N_{t_j} - N_{t_{j-1}+dt_{j-1}} = 0) \cdot P(N_{t_j+dt_j} - N_{t_j} = 1)
\]

\[
= \prod_{j=1}^{k} e^{-\lambda(t_j-t_{j-1})} \lambda dt_j = e^{-\lambda t_k} \lambda^k dt_1 \cdots dt_k,
\]

so for all \( t_{k+1} > t_k \) we have

\[
P\left( t_{k+1} < T_{k+1} \leq t_{k+1} + dt_{k+1} \left| \bigcap_{j=1}^{k} t_j < T_j \leq t_j + dt_j \right. \right) = \lambda e^{-\lambda(t_{k+1}-t_k)} dt_{k+1}.
\]

This shows that the waiting time \( T_{k+1} - T_k \) has distribution \( \text{Exponential}(\lambda) \) and is independent of previous waiting times. \( \blacksquare \)

### 5 Poisson Point Processes

**Definition 3** A Poisson Point Process (P.P.P.) with intensity measure \( \mu \) on \( (S, \mathcal{S}) \) is a collection of random variables \( N(B, \omega) \), \( B \in \mathcal{S}, \omega \in \Omega \) defined on a probability space \( (\Omega, \mathcal{F}, P) \) such that:

1. \( N(B) = N(B, \omega) \), \( B \in \mathcal{S}, \omega \in \Omega \);
2. \( N(\cdot, \omega) \) is a non-negative integer or \( \infty \)-valued measure on \( (S, \mathcal{S}) \) for each \( \omega \in \Omega \);
3. \( N(B, \cdot) \) is a r.v. with Poisson(\( \mu(B) \)) distribution:

\[
P(N(B) = k) = \frac{e^{-\mu(B)}(\mu(B))^k}{k!} \text{ for all } B \in \mathcal{S};
\]

4. If \( B_1, B_2, \ldots \) are disjoint sets then \( N(B_1, \cdot), N(B_2, \cdot), \ldots \) are independent random variables.

We can think of a Poisson point process as a generalization of a Poisson process dealing with the observation of events at points in space, \( S \), rather than points in time, \( [0, \infty) \).
For example, if we let $T_k$ denote the time of the $k$-th event in a Poisson process with rate $\lambda$, then $N(B) = \#\{k \in \mathbb{N} : T_k \in B\}$ defines a Poisson point process on $(S, \mathcal{S}, \mu)$ where $S = [0, \infty)$, $\mathcal{S}$ is the Borel $\sigma$-algebra, and $\mu$ has constant density $\lambda$ with respect to Lebesgue measure.

Similarly, if for $k \in \mathbb{N}$ we define $T_{-k} = -T'_k$ where $T'_k$ is the time of the $k$-th event in a second, independent Poisson process of rate $\lambda$, then $N(B) = \#\{k \in \mathbb{Z} : T_k \in B\}$ defines a Poisson point process on $(S, \mathcal{S}, \mu)$ where $S = \mathbb{R}$, $\mathcal{S}$ is the Borel $\sigma$-algebra, and $\mu$ has constant density $\lambda$ with respect to Lebesgue measure.

**Theorem 4** There is a Poisson point process on $(S, \mathcal{S}, \mu)$ if $\mu$ is $\sigma$-finite.

**Proof:** We will prove the claim for the case $\mu(S) < \infty$; the general case follows easily from this one by piecing together countably many independent Poisson point processes on finite-measure subsets of $S$.

Let $X_1, X_2, \ldots$ be i.i.d. $S$-valued random variables with distribution $\mu/\mu(S)$, and let $N(S)$ be a random variable of distribution Poisson($\mu(S)$) that is independent of the $X_i$'s. Then define $N(B) = \#\{i \leq N : X_i \in B\}$. Clearly $N$ is a measure for every $\omega \in \Omega$, and conditions 1 and 2 of the definition follow from the thinning property of Poisson distributions (Exercise 2.6.12 in Durrett3.)

**References**