1 Prerequisites

Definition of a random variable

2 Summary

In this section we have three goals: To state formally the notion of convergence in probability; to state the weak law of large numbers in terms of convergence in probability; and to review informally the various other notions of convergence for random variables.

3 Notions of Convergence for Random Variables

**Definition 1** Given a sequence of r.v’s $X_n$, $n \in \mathbb{N}$, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. If for all $\epsilon > 0$,

$$\lim_{n \to \infty} \mathbb{P}(\vert X_n - X_0 \vert > \epsilon) = 0,$$

then we say $X_n$ converges in probability to $X_0$, denoted as $X_n \xrightarrow{p} X_0$.

With this definition, we can now state the weak law of large numbers (WLLN):

**Theorem 2 (Weak Law of Large Numbers)** Let $X, X_1, X_2, \ldots$ be i.i.d. with $\mathbb{E}|X| < \infty$. Then

$$\frac{1}{n} \sum_{i=1}^{n} X_i \xrightarrow{p} \mathbb{E}(X).$$
To put this in perspective, we now review other notions of convergence of r.v.’s:

**Pointwise Convergence:** \( X_n(\omega) \rightarrow X(\omega) \) for all \( \omega \in \Omega \). This is a very strong notion: too strong for many purposes.

**Almost Sure Convergence:** We say \( X_n \overset{a.s.}{\rightarrow} X \) if \( X_n(\omega) \rightarrow X(\omega) \) for all \( \omega \not\in N \), with \( P(N) = 0 \), or equivalently \( P\{\omega \in \Omega : X_n(\omega) \rightarrow X(\omega) \text{ as } n \rightarrow \infty\} = 1 \).

**Convergence in \( L^p \) (\( p \geq 1 \)):** We say \( X_n \overset{L^p}{\rightarrow} X \) if \( \|X_n - X\|_p \rightarrow 0 \), i.e. \( \lim_{n \rightarrow \infty} E|X_n - X|^p = 0 \).

**Convergence in Distribution:** (Not really a notion of convergence of r.v.) A notion of convergence of a probability distribution on \( \mathbb{R} \) (or more general space). We say \( X_n \overset{d}{\rightarrow} X \) if \( P(X_n \leq x) \rightarrow P(X \leq x) \) for all \( x \) at which the RHS is continuous.

This weak convergence appears in the central limit theorem.

**Example 3** (See Durrett [1], pg. 84) \( X_n \overset{d}{\rightarrow} X \iff Ef(X_n) \rightarrow Ef(X) \) for all bounded and continuous function \( f \).

**Properties in Common for** \( \overset{p}{\rightarrow}, \overset{p.w.}{\rightarrow}, \overset{a.s.}{\rightarrow}, \overset{L^p}{\rightarrow} \):

a) \( X_n \rightarrow X, Y_n \rightarrow Y \implies X_n + Y_n \rightarrow X + Y, X_nY_n \rightarrow XY \).

b) \( X_n \rightarrow X \iff (X_n - X) \rightarrow 0 \) (useful and common reduction).

c) For all of \( \overset{p}{\rightarrow}, \overset{a.s.}{\rightarrow}, \text{ and } \overset{L^p}{\rightarrow} \) the limit \( X \) is unique up to a.s. equivalence.

d) Cauchy sequences are convergent (completeness). (Need a metric to metrize \( \overset{p}{\rightarrow} \), but that is easily provided. See text.)

**Theorem 4** The following property holds among the types of convergence.
Proof: (*) can be proved by Chebyshev’s inequality (with usually $p = 2$):

$$
P(|X_n - X| > \epsilon) \leq \frac{E(|X_n - X|^p)}{\epsilon^p}
$$

(**) is proved in the text. [1]

Example 5 (Moving blip) (An example showing that almost sure convergence is a stronger condition than convergence in probability.) Let $r = 1/2\pi$. Equipped $\{re^{\theta} \in \mathbb{C} \mid \theta \in \mathbb{R} \}$ with a probability $\mathbb{P}$ such that $\mathbb{P}\{re^{\theta} \in \mathbb{C} \mid 0 \leq \theta_1 \leq \theta \leq \theta_2 \leq 2\pi\} = (\theta_2 - \theta_1)/2\pi$. Let $\phi_n = \sum_{k=1}^n \pi/k$, and for $n \geq 1$, $A_n = \{re^{i\theta} \in \mathbb{C} \mid \phi_{n-1} \leq \theta \leq \phi_n\}$, $X_n = 1_{A_n}$. Then for $\epsilon > 0$, $\mathbb{P}(|X_n| > \epsilon) = \pi/n \to 0$, but $\{\zeta \mid X_n(\zeta) \to 0\} = \emptyset$.

Example 6 Suppose that $X_1, X_2, \ldots$ are r.v.'s that have mean 0, have finite variances, and are uncorrelated. Let $S_n = X_1 + \cdots + X_n$. If $\sum_{k=1}^{\infty} E(X_k^2) < \infty$, then $S_n$ converges in $L^2$ to a limit $S_\infty$, hence $S_n \overset{p}{\to} S_\infty$, i.e. $\lim_{n \to \infty} \mathbb{P}(|S_n - S_\infty| > \epsilon) = 0$ for all $\epsilon > 0$.

Proof: Take $m > n$:

$$
E(S_m - S_n)^2 = E\left(\sum_{k=n+1}^{m} X_k\right)^2 = \sum_{k=n+1}^{m} E(X_k^2) \to 0
$$

as $m, n \to \infty$. Therefore $S_\infty$ exists because $L^2$ is complete.

Example 7 If the $X_n$ are independent (or more generally, martingale distributions), then $S_n \overset{a.s.}{\to} S_\infty$.

The proof of this fact is deferred.

Example 8 (Stout’s Almost Sure Convergence) There are examples of uncorrelated sequences with $\sum_n X_n^2 < \infty$ where a.s. convergence fails.

References