1 Prerequisites

Definition of the characteristic function, Properties of the characteristic function, Inversion formula

2 Summary

In this section, some examples of the concept of characteristic function are given. See section 2.3 of [1].

3 Examples

3.1 Exponential Distribution and a Variation

Let $X$ have exponential distribution with parameter $\lambda > 0$, i.e. $X$ has density $f_X(x) = \lambda e^{-\lambda x}1(x > 0)$. Then we can calculate the characteristic function of $X$:

$$\varphi_X(t) = \int e^{itx} \mathbb{P}_X(dx) = \int_0^\infty e^{itx} \lambda e^{-\lambda x} dx = \lambda \int_0^\infty e^{(it-\lambda)x} dx = \lambda \frac{e^{(it-\lambda)x}|_0^\infty}{it - \lambda} = \frac{\lambda}{\lambda - it}$$

since $|e^{(it-\lambda)x}| = e^{-\lambda x} \to 0$ as $x \to \infty$.

Note that, if the characteristic function of a random variable $X$ is real, then $X \overset{d}{=} -X$ and $\varphi_X(t) = \mathbb{E}[e^{-itX}] = \varphi_{-X}(t)$. So:

$$X \overset{d}{=} -X \iff \varphi_X(t) = \varphi_{-X}(t) \iff \varphi_X(t) = e^{-it\lambda} \iff \varphi_X(t) \in \mathbb{R}$$

Consider the characteristic function of $X - \hat{X}$, where $\hat{X}$ has exponential distribution with parameter $\lambda$ and $X, \hat{X}$ are independent.

$$\varphi_{X-\hat{X}}(t) = \varphi_X(t) \varphi_{\hat{X}}(t) = \varphi_X(t) \varphi_{\hat{X}}(-t) = \frac{\lambda}{(\lambda - it)(\lambda + it)} = \frac{\lambda^2}{\lambda^2 + t^2}$$
Take now \( \lambda = 1 \) and the density of \( X - \hat{X} \) which is:

\[
f_{X-\hat{X}}(x) = \frac{1}{2} e^{-|x|}
\]

This leads to a classic integral identity

\[
\int e^{itx} \frac{1}{2} e^{-|x|} dx = \frac{1}{1 + t^2}
\]

Since \( \int \varphi_{X-\hat{X}}(t) dt = \int \frac{1}{1+t^2} dt < \infty \), the inversion formula gives us:

\[
\frac{1}{2\pi} \int e^{-itx} \frac{1}{1 + t^2} dt = \frac{1}{2} e^{-|x|}
\]

(1)

Take \( x = 0 \). Another classic integral is

\[
\int \frac{1}{1 + t^2} dt = \pi
\]

### 3.2 The Cauchy Distribution

Let \( Y \) have Cauchy distribution, i.e. \( Y \) has density \( f_Y(x) = \frac{1}{\pi(1+x^2)} \). Notice that \( \mathbb{E}|Y| = \infty \) and the Cauchy distribution has a heavy tail compared to other distributions. Using (1), the characteristic function of \( Y \) is

\[
\varphi_Y(t) = \int e^{itx} f_Y(dx) = \int e^{itx} \frac{1}{\pi(1+x^2)} dx = e^{-|t|}.
\]

Now let \( Y_1, \ldots, Y_n \) be i.i.d. with the Cauchy distribution and \( S_n = Y_1 + \cdots + Y_n \). The characteristic function of \( \frac{S_n}{n} \) is

\[
\varphi_{\frac{S_n}{n}}(t) = \varphi_{S_n}(\frac{t}{n}) = \prod_{i=1}^n \varphi_{Y_i}(\frac{t}{n}) = (e^{-\frac{|t|}{n}})^n = e^{-|t|} = \varphi_Y(t)
\]

Hence \( \frac{S_n}{n} \overset{d}{\rightarrow} Y \).

This is the amazing property of the Cauchy distribution: the average of \( n \) independent Cauchy variables has the same invariant Cauchy distribution. There is obviously no law of large numbers effect here! If we hadn’t known that \( \mathbb{E}|Y| = \infty \), we would have been able to conclude that now because of the strong law of large numbers.
3.3 Common distributions and their characteristic functions

Here we list down some common distributions and their characteristic functions. See the table.

<table>
<thead>
<tr>
<th>Distribution</th>
<th>Density</th>
<th>Support</th>
<th>Characteristic Function</th>
</tr>
</thead>
<tbody>
<tr>
<td>Standard Normal</td>
<td>$\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$</td>
<td>$\mathbb{R}$</td>
<td>$e^{-\frac{t^2}{2}}$</td>
</tr>
<tr>
<td>Standard Uniform</td>
<td>1</td>
<td>[0, 1]</td>
<td>$\frac{e^{it}-1}{it}$</td>
</tr>
<tr>
<td>Exponential</td>
<td>$e^{-x}$</td>
<td>(0, $\infty$)</td>
<td>$\frac{1}{1-it}$</td>
</tr>
<tr>
<td>Double Exponential</td>
<td>$\frac{1}{2}e^{-</td>
<td>x</td>
<td>}$</td>
</tr>
<tr>
<td>Cauchy</td>
<td>$\frac{1}{\pi} \frac{1}{1+x^2}$</td>
<td>$\mathbb{R}$</td>
<td>$e^{-</td>
</tr>
<tr>
<td>Triangular</td>
<td>$1 -</td>
<td>x</td>
<td>$</td>
</tr>
</tbody>
</table>

References