1 Prerequisites

Parseval’s identity, Fubini’s Theorem, convolution, weak convergence, almost sure convergence

2 Introduction

This topic covers section 2.3 of [1].

Recall that the characteristic function of a random variable $X$ with distribution $\mathbb{P}$ is defined as follows:

$$\varphi_X(t) = \mathbb{E}e^{itX} = \int e^{ix}\mathbb{P}(dx), \quad t \in \mathbb{R}. \quad (1)$$

Given the distribution of $X$, we can compute the characteristic function of $X$ using equation (1). But if the characteristic function $\varphi_X(t)$ is known, how can the distribution of $X$ be recovered?

The Uniqueness Theorem tells us that there is exactly one distribution of $X$ with the given characteristic function $\varphi_X(t)$. We will derive an inversion formula that explicitly tells us this distribution in terms of $\varphi_X(t)$.

Below we will first work out the characteristic function of $N(0, 1)$ explicitly as it will be used in the derivation of inversion formula.

3 Characteristic function of $N(0, 1)$

Suppose that $X \sim N(0, 1)$. A key identity is:

$$\varphi_X(t) = \mathbb{E}e^{itX} = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}e^{-x^2/2}e^{itx}dx = e^{-t^2/2} \quad (2)$$
Here are four ways to prove this identity (Section 2.3 of Durrett gives two more):

1. By real analysis, for real $\theta$,
   \[
   \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} e^{\theta x} dx = e^{\theta^2/2}
   \] (3)
   After confirming that the identity holds for $\theta \in \mathbb{R}$, extend it for $\theta \in \mathbb{C}$ by standard complex analysis techniques such as analytic continuation. Equation 2 follows by substituting $\theta = it$.

2. Use complex line integration, i.e. integrate real part and imaginary part separately.

3. Use differential equations: Check that both sides of equation (2) satisfy the same differential equation when differentiated with respect to $t$. They agree at $t = 0$, so they must agree at all $t$.

4. Use the Central Limit Theorem. This is kind of backwards, since we usually use characteristic functions to prove the Central Limit Theorem. However, the Central Limit Theorem can be proved via Lindeberg’s Theorem, so circular reasoning is not present.

Take $X_1, X_2, \ldots$ i.i.d. which take on values of $-1$ and $+1$ with $1/2$ probability each. Let $S_n = X_1 + \cdots + X_n$.

By the Central Limit Theorem, $S_n/\sqrt{n} \xrightarrow{d} N(0, 1)$. Since $x \rightarrow e^{itx}$ is a bounded continuous function, $\mathbb{E} e^{itS_n/\sqrt{n}} \rightarrow \mathbb{E} e^{itZ}$, where $Z \sim N(0, 1)$. The left hand side can be computed explicitly:

\[
\mathbb{E} e^{itS_n/\sqrt{n}} = \left( \mathbb{E} e^{itX_1/\sqrt{n}} \right)^n \quad \text{[because $X_i$’s are i.i.d.]} \quad (4)
\]
\[
= \left( \frac{1}{2} e^{it/\sqrt{n}} + \frac{1}{2} e^{-it/\sqrt{n}} \right)^n
\]
\[
= \left( \cos(t/\sqrt{n}) \right)^n
\]
\[
= \left( 1 - \frac{t^2}{2n} + O \left( \frac{t^4}{n^2} \right) \right)^n \quad \text{[as $n \rightarrow \infty$]} \quad (7)
\]
\[
\rightarrow e^{-t^2/2} \quad \text{[Fact: $(1 + x/n)^n \rightarrow e^x$ if $x_n \rightarrow x$]}
\]

where, in the fourth line, we used the fact that as $n \rightarrow \infty$, $\cos(t/\sqrt{n})$ can be approximated by its Taylor expansion around $t/\sqrt{n} = 0$.

By simple properties of characteristic functions, the characteristic function of a normal random variable with mean $\mu$ and variance $\sigma$ can be written as:

\[
\varphi_{\mu + \sigma Z}(t) = e^{it\mu - \sigma^2 t^2/2},
\] (9)

where $Z \sim N(0, 1)$.
4 Derivation of the inversion formula

Parseval’s identity gives us:

$$E[\varphi_X(Y)] = E[\varphi_Y(X)]$$  \hspace{1cm} (10)

Apply (10) with $Y$ distributed as $Q = N(0, \sigma^2)$:

$$\int \varphi_X(t) \frac{1}{\sqrt{2\pi\sigma^2}} e^{-t^2/2\sigma^2} dt = \int e^{-\sigma^2 x^2/2} \mathbb{P}(dx).$$  \hspace{1cm} (11)

The left-hand side looks fine, but we would like to decipher the right-hand side in terms of $\mathbb{P}$. Interpret the right-hand side probabilistically.

The following observation can help us: If $X$ and $Y$ are independent random variables with $X$ having distribution $\mathbb{P}$ and $Y$ having distribution given by density $f_Y$, then $X + Y$ has a density described by the following convolution formula (by Fubini’s Theorem):

$$f_{X+Y}(z) = \int_{-\infty}^{\infty} f_Y(z - x) \mathbb{P}(dx),$$  \hspace{1cm} (12)

Match the right-hand side of (11) with (12). Let $Y = Z/\sigma$ be distributed as $N(0, 1/\sigma^2)$ with density

$$f_Y(y) = f_{Z/\sigma}(y) = \frac{\sigma}{\sqrt{2\pi}} e^{-y^2\sigma^2/2}$$  \hspace{1cm} (13)

By noting that $f_{Z/\sigma}(x) = f_{Z/\sigma}(-x)$, (12) becomes:

$$f_{X+Z/\sigma}(0) = \int_{-\infty}^{\infty} f_{Z/\sigma}(x) \mathbb{P}(dx).$$  \hspace{1cm} (14)

Now interpret the right-hand side of (11) as the density of $X + Z/\sigma$ evaluated at 0 times the constant $\frac{\sqrt{2\pi}}{\sigma}$.

Combining (13), (11), and (14):

$$\int \varphi_X(t) \frac{1}{\sqrt{2\pi\sigma^2}} e^{-t^2/2\sigma^2} dt = \frac{\sqrt{2\pi}}{\sigma} \int f_{Z/\sigma}(x) \mathbb{P}(dx) = \frac{\sqrt{2\pi}}{\sigma} f_{X+Z/\sigma}(0)$$  \hspace{1cm} (15)

$$f_{X+Z/\sigma}(0) = \frac{1}{2\pi} \int \varphi_X(t) e^{-t^2/2\sigma^2} dt$$  \hspace{1cm} (16)

(Note that the calculations above worked due to two favorable properties of the normal distribution: first, its characteristic function is proportional to a probability density; and second, its density is symmetric around 0.)
As a matter of aesthetics, give $\sigma$ its usual interpretation as the standard deviation by replacing $\sigma$ with $1/\sigma$:

$$f_{X+\sigma Z}(0) = \frac{1}{2\pi} \int \varphi_X(t)e^{-t^2\sigma^2/2}dt$$

To get the value of $f_{X+\sigma Z}(x)$ at other values of $x$ other than 0, note that $f_{X+\sigma Z}(x) = f_{X-x+\sigma Z}(0)$, which gives the final formula for the density of $X + \sigma Z$:

$$f_{X+\sigma Z}(x) = \frac{1}{2\pi} \int \varphi_X(t)e^{-itx}e^{-t^2\sigma^2/2}dt$$

In general, the distribution of $X$ could be rough and not have a density. Nonetheless, $X + \sigma Z$ does have a density for all $\sigma > 0$, which is made explicit in terms of $\varphi_X(t)$.

Now try to pinpoint the distribution of $X$ itself. As $\sigma \to 0$, $X + \sigma Z \xrightarrow{a.s.} X$, so also in distribution. Thus, for all bounded continuous functions $g$,

$$\mathbb{E}g(X) = \lim_{\sigma \to 0} \mathbb{E}g(X + \sigma Z) = \lim_{\sigma \to 0} \int g(x)f_{X+\sigma Z}(x)dx$$

Therefore, $\varphi_X(t)$ determines the distribution of $X$ as the weak limit of the distributions of $X + \sigma Z$ as $\sigma \to 0$.

We can do more when $\varphi_X(t)$ is absolutely integrable, that is, if $\int |\varphi_X(t)|dt < \infty$. This condition basically imposes some smoothness constraints on the distribution of $X$. By the Dominated Convergence Theorem, the limit and the expectation can be exchanged in equation (19):

$$\mathbb{E}g(X) = \lim_{\sigma \to 0} \int g(x)f_{X+\sigma Z}(x)dx = \int g(x)f_X(x)dx,$$

where $X$ now has an explicit continuous density:

$$f_X(x) = \frac{1}{2\pi} \int \varphi_X(t)e^{-itx}dt$$

Durrett gives a general inversion formula regardless of whether $X$ has a density or not. If $a < b$, then

$$\lim_{T \to \infty} \frac{1}{2\pi} \int_{-T}^{T} \frac{e^{-ita} - e^{-itb}}{it}\varphi_X(t)dt = \mathbb{P}(a,b) + \frac{1}{2}\mathbb{P}\{\{a,b\}\}$$

Some other good information about the distribution of $X$ can be obtained from the characteristic function, most usefully bounds on tail probabilities, for example see Kallenberg [2, Chap. 5.1].
References
