1 Prerequisites

Characteristic function, infinitely divisible function, levy process

2 Summary

This topic gives definitions of stable-related distributions and a theorem on $\alpha$-stable levy process. See Section 2.7 of [1]. We also refer to [2] for some of the advanced materials.

3 Stable laws

**Definition 1** $Y$ is stable or has a stable law if, for each $n \in \mathbb{N}$, $\exists a_n$ and $b_n$ so that for i.i.d $Y_1, \ldots, Y_n$ that have the same distribution of $Y$, then

$$\frac{Y_1 + \ldots + Y_n - b_n}{a_n} \overset{d}{=} Y$$

Moreover, stable law distributions have the following characteristic functions:

$$\phi(t) = \exp(itc - b|t|^{\alpha}(1 + ik\omega_{\alpha}(t)\text{sgn}(t)))$$

Where $-1 \leq \kappa \leq 1$, $0 < \alpha \leq 2$ (strictly greater than 0) and

$$\omega_{\alpha}(t) = \begin{cases} \tan(\frac{\pi \alpha}{2}) & \text{if } \alpha \neq 1 \\ (\frac{2}{\pi}) \log |t| & \text{if } \alpha = 1 \end{cases}$$

The distribution is said to have index $\alpha$.

$Y$ is said to be strictly stable if it is possible to let $b_n = 0, \forall n \in \mathbb{N}$. $Y$ is said to be symmetric stable if it is stable and $Y \overset{d}{=} -Y$.

From the definition, it is straightforward that stable r.v.’s are $\infty$-divisible.
Definition 2  A stable $Y$ is called $\alpha-$stable, $\alpha \in (0,2]$, whenever $\forall n \in \mathbb{N}$

$$
\sum_{i=1}^{n} Y_i \overset{d}{=} n^{1/\alpha} Y + b_n \text{ some constant } b_n \in \mathbb{R}^d.
$$

$Y$ is called strictly $\alpha-$stable if $b_n = 0$.

For $Y$ non-trivial (strictly) stable, there exists a unique constant $\alpha \in (0,2]$ such that $Y$ is $\alpha-$stable (strictly).

Definition 3  An rcll (right continuous left limit) process $X$ in $\mathbb{R}^d$ with stationary independent increments and $X_0 = 0$ is called a Lévy process.

Definition 4  A Levy process $X$ with $X(1)$ (strictly) $\alpha-$stable is called a (strictly) $\alpha-$stable Levy motion.

Theorem 5  Let $X$ be a non-trivial Levy process in $\mathbb{R}$ with generating triplet $(A,\gamma,\nu)$. Then $X$ is $\alpha-$stable for some $\alpha > 0$ iff exactly one of these conditions holds:

1. $\alpha = 2$ and $\nu = 0$.
2. $\alpha \in (0,2)$, $A = 0$, and $\nu(dx) = (c_+ 1_{(0,\infty)}(x) + c_- 1_{(-\infty,0)}(x)) |x|^{-(\alpha+1)}dx$ on $\mathbb{R}$ for some $c_+, c_- \geq 0$.

Proof: Suppose the generating triplet of $Y = X(1)$ is $(A,\gamma,\nu)$. We know $Y$ is $\alpha-$stable iff $Y^{*r^\alpha} \overset{d}{=} rY + c$ for $t > 0$ and some constant $c$. Since the characteristics of $Y^{*r^\alpha}$ and $rY + c$ are $r^{\alpha}(A,\gamma,\nu)$ and $(r^2A, r\gamma + c, \nu \circ S^{-1}_r)$ respectively, where $S_r : x \mapsto rx$ for any $r > 0$. It follows that $X(t)$ is $\alpha-$stable iff $r^\alpha A = r^2 A$ and $r^\alpha \nu = \nu \circ S^{-1}_r$ for all $r > 0$. Thus, $A = 0$ when $\alpha \neq 2$. Writing $F(x) = \nu[0,\infty]$ or $\nu(-\infty,0]$, $r^\alpha \nu = \nu \circ S^{-1}_r$ implies $r^\alpha F(rx) = F(x)$ for all $r, x > 0$, and so $F(x) = x^{-\alpha} F(1)$. The condition $\int(x^2 \wedge 1) \nu(dx) < \infty$ implies $F(1) < \infty$ and when $\alpha = 2$, we have $F(1) = 0$.

4 Examples

Example 6 (Normal Distribution)  The Normal distribution is a stable distribution with characteristic function index $\alpha = 2$. Notice that implies $\omega_\alpha(t) = 0$. Realize that this Normal distribution has mean $c$, and variance $2b$. 

Example 7 (Cauchy Distribution) The Cauchy distribution is a stable distribution with characteristic function index $\alpha = 1$ and $\kappa = 0$.

Example 8 (Another known stable law) The following density is a stable law with characteristic function index $\alpha = \frac{1}{2}$, $\kappa = 1$, $c = 0$, $b = 1$, for $x \geq 0$.

$$(2\pi x^3)^{\frac{1}{2}} \exp\left(\frac{1}{2x}\right)$$

If $Z$ is the Standard Normal Distribution (mean 0 with variance 1), $\frac{1}{Z^2}$ has the above density.

References
