1 Prerequisites

Exchangeable sequence of random variables, exchangeable sigma-field, Martingale Convergence Theorem

2 Summary

This topic uses martingale convergence theorem to prove Hewitt-Savage 0-1 Law.

3 Hewitt-Savage 0 - 1 Law

Theorem 1 (Hewitt-Savage 0 - 1 Law) If $X_1, X_2, \ldots$ is i.i.d., then every event in the exchangeable sigma-field $\mathcal{E}_\infty$ has probability 0 or 1.

(Compare this with Kolmogorov’s 0 – 1 Law.)

Recall that if $X_1, X_2, \ldots$ are i.i.d. and exchangeable (i.e.,

$$(X_1, X_2, \ldots, X_n) \overset{d}{=} (X_{\pi(1)}, X_{\pi(2)}, \ldots, X_{\pi(n)})$$

for all permutations $\pi$ on $n$ elements) and $\mathbb{E}|X_1| < \infty$, then $(S_n/n, \mathcal{E}_n)_{n \geq 1}$ is a reversed martingale. If $S_n/n = \mathbb{E}(X_1|\mathcal{E}_n)$, then this is obvious because $\mathcal{E}_n \downarrow$.

Exchangeability implies that

$$\mathbb{E}(X_1|\mathcal{E}_n) = \mathbb{E}(X_k|\mathcal{E}_n) \quad \text{for every } 1 \leq k \leq n.$$

This is because

$$(X_1, f(X_1, \ldots, X_n, X_{n+1}, X_{n+2}, \ldots))
\overset{d}{=} (X_k, f(X_k, \ldots, X_{k-1}, X_1, X_{k+1}, \ldots, X_n, X_{n+1}, X_{n+2}, \ldots))
= (X_k, f(X_1, X_2, \ldots, X_n, X_{n+1}, X_{n+2}, \ldots))$$
Check the definition of the claim:

\[ n \mathbb{E}(X_1 | \mathcal{E}_n) = \mathbb{E}(S_n | \mathcal{E}_n) = S_n \text{ since } S_n \subseteq \mathcal{E}_n. \]

To prove the theorem, show that for every event \( F \in \sigma(X_1, \ldots, X_n) \), \( \mathbb{P}(F | \mathcal{E}_\infty) = \mathbb{P}(F) \). This says that \( \sigma(X_1, \ldots, X_n) \) is independent of \( \mathcal{E}_\infty \). Let \( n \to \infty \) and learn that \( \sigma(X_1, X_2, \ldots) \) is independent of \( \mathcal{E}_\infty \). This implies that \( \mathcal{E}_n \) is independent of \( \mathcal{E}_\infty \), which leads to the result of the 0−1 Law.

**Proof:** By the Martingale Convergence Theorem,

\[ \mathbb{P}(F | \mathcal{E}_\infty) = \lim_{n \to \infty} \mathbb{P}(F | \mathcal{E}_n) \]

Let \( F = \{ X_1 \in \hat{F} \} \) for some \( \hat{F} \subseteq \mathbb{R} \). Then,

\[
\begin{align*}
\mathbb{P}(F | \mathcal{E}_n) &= \mathbb{E}(1_{\hat{F}(X_1)} | \mathcal{E}_n) \\
&= \mathbb{E}(1_{\hat{F}(X_k)} | \mathcal{E}_n) \\
&= \mathbb{E} \left( \frac{1}{n} \sum_{k=1}^{n} 1_{\hat{F}(X_k)} | \mathcal{E}_n \right) \\
&= \frac{1}{n} \sum_{k=1}^{n} 1_{\hat{F}(X_k)} \\
&\to \mathbb{E}(1_{\hat{F}(X_k)}) \\
&= \mathbb{P}(X_1 \in \hat{F}) \\
&= \mathbb{P}(F).
\end{align*}
\]

This proves the case \( F \in \sigma(X_1) \). To deal with the case \( F \in \sigma(X_1, X_2) \), do the same thing. Let \( F = \{(X_1, X_2) \in \hat{F} \} \) for some \( \hat{F} \subseteq \mathbb{R}^2 \) and \( \varphi(X_1, X_2) = 1((X_1, X_2) \in \hat{F}) \).

\[
\begin{align*}
\mathbb{P}(F | \mathcal{E}_n) &= \mathbb{E}(\varphi(X_1, X_2) | \mathcal{E}_n) \\
&= \mathbb{E}(\varphi(X_i, X_j) | \mathcal{E}_n) \text{ for } i \neq j \\
&= \mathbb{E} \left( \frac{1}{n(n-1)} \sum_{1 \leq i \neq j \leq n} \varphi(X_i, X_j) | \mathcal{E}_n \right) \\
&= \frac{1}{n(n-1)} \sum_{1 \leq i \neq j \leq n} \varphi(X_i, X_j).
\end{align*}
\]
Consider $\varphi(X_i, X_j) = f(X_i)g(X_j)$, i.e. $\hat{F}$ is rectangular; then

\[
\mathbb{P}(F \mid \mathcal{E}_n) = \frac{1}{n(n-1)} \sum_{1 \leq i \neq j \leq n} f(X_i)g(X_j) \\
= \frac{1}{n(n-1)} \left( \sum_{1 \leq i,j \leq n} f(X_i)g(X_j) - \sum_{i=1}^{n} f(X_i)g(X_i) \right) \\
= \frac{1}{n(n-1)} \left( \sum_{i=1}^{n} f(X_i) \sum_{i=1}^{n} g(X_i) - \sum_{i=1}^{n} f(X_i)g(X_i) \right) \\
\overset{a.s.}{\to} \mathbb{E}f(X_1)\mathbb{E}g(X_2) \\
= \mathbb{E}\varphi(X_1, X_2) \\
= \mathbb{P}(F).
\]

To finish, for $\hat{F} \in \mathcal{B}(\mathbb{R}^2)$, use the $\pi - \lambda$ theorem. Similarly for $\hat{F} \in \mathcal{B}(\mathbb{R}^k)$, $k \geq 3$.

So far, we’ve shown that

\[
\mathbb{P}(F \mid \mathcal{E}_\infty) = \mathbb{P}(F) \text{ for all } F \in \sigma(X_1, \ldots, X_n),
\]

i.e. $\mathcal{E}_\infty$ is independent of $\sigma(X_1, \ldots, X_n)$. Similarly as in the proof of Kolmogorov’s 0-1 Law, we can learn that $\mathcal{E}_\infty$ is independent of $\sigma(X_1, X_2, \ldots)$ by sending $n \to \infty$. Thus $\mathcal{E}_\infty$ is independent of itself, which completes the proof. \(\blacksquare\)

4 Reference