1 Prerequisites

Characteristic function, convolution, Lévy-Khintchine theorem

2 Summary

Infinitely divisible defines a special class of distributions, which has Lévy-Khintchine representation and some other properties.

3 Definition

**Definition 1** A distribution $F$ is infinitely divisible if for each $n$, $F$ has such a convolution $n$th root, i.e.

$$F_1 = F_{1/n} * F_{1/n} * \ldots * F_{1/n} \quad (1)$$

Therefore its characteristic function $\phi_1$ has an $n$th root. That is, $F$ is the distribution of a sum of $n$ i.i.d. random variables.

4 Major Facts of Infinitely Divisible Distributions

There are a few major results of infinitely divisible distributions.

**Fact 1** Every infinitely divisible law $F$ has a unique $n$th root $F_{1/n}$, and indeed there is a unique weakly continuous convolution semigroup $(F_t, t \geq 0)$ such that $F_1 = F$.

This immediately implies the following:
**Fact 2** Every infinitely divisible law $F$ is associated with a process $(X_t, t \geq 0)$ with SII (stationary independent increments) and right continuous paths, and $X_t \sim F_t$.

**Fact 3** There is an explicit description (the Lévy-Khintchine representation) for the characteristic function of the most general infinitely-divisible law.

**Fact 4** The collection of infinitely divisible distributions is precisely the collection of all possible weak limits of

$$S_n = \sum_j X_{nj}$$

where $X_{nj}$ is the $(n,j)\text{th}$ entry of a UAN (uniform asymptotic negligibility) array of (not necessarily identically distributed) random variables that are independent within the rows.

The Lévy-Khintchine formula yields the following corollary.

**Corollary 1** The characteristic function of an infinitely divisible distribution never vanishes and therefore the convolution $n\text{th}$ roots are unique.

**Fact 5** There exists a probability distribution for $X$ with $\phi_X(t) = e^{-|t|^\alpha}$ if and only if $0 \leq \alpha \leq 2$.

Existence is easy for the case $0 \leq \alpha \leq 1$ because, for these values, the function $t \mapsto e^{-|t|^\alpha}$ is convex; Polya’s criterion tells us then that this function is a characteristic function for some random variable.

The case where $1 \leq \alpha \leq 2$ is harder.

## 5 Reference

