1 Prerequisites

Lévy measure, characteristic function, infinitely divisible distribution, compound poisson distribution

2 Summary

This topic gives proof of Lévy-Khintchine Theorem and reveals the connection between Lévy-Khintchine exponent and infinitely divisible distributions. Examples are also given including point mass, normal and poisson distribution.

3 Lévy-Khintchine Theorem

For a Lévy measure $L$, $\sigma^2 \geq 0$, $c \in \mathbb{R}$, define the Lévy-Khintchine exponent in the following way:

$$\Psi_{L,\sigma^2,c}(t) = \int \left( e^{itx} - 1 - it\tau(x) \right) L(dx) - \frac{1}{2} \sigma^2 t^2 + itc,$$

where $\tau(x)$ is the truncation function defined by $\tau(x) = x1_{|x|\leq1} + 1_{x>1} - 1_{x<1}$.

**Theorem.**

1. $e^{\Psi(t)}$ is an infinitely divisible characteristic function.
2. $e^{\Psi(t)}$ determines $L, \sigma^2, c$ uniquely.

Note: the converse of 1 is also true.

**Examples.**
(a) A point mass $\delta_c$ at $c$. Its characteristic function is $e^{ict}$, and $ict = \Psi_{0,0,c}(t)$.

(b) A normal distribution $N(c, \sigma^2)$. Its characteristic function is $e^{ict-\sigma^2t^2/2}$ and it is easy to see that $\Psi(t) = ict - \sigma^2t^2/2$ corresponds to $(0, \sigma^2, c)$.

(c) Let $N$ be a Poisson random measure. For each $f \geq 0$, 
$$E \left( e^{-\theta \int fdN} \right) = \exp \left( \int (e^{-\theta f(x)} - 1) \, \mu(dx) \right)$$ (see compound poisson distribution).

If $\mu$ is bounded measure, take $\theta = -it$, 
$$E \left( e^{it \int fdN} \right) = \exp \left( \int (e^{itf(x)} - 1) \, \mu(dx) \right).$$

Let $L(dy) = \mu\{x : f(x) \in dy\}$ (restricted to $\{0\}^c$). That is, $L(B) = \mu(f^{-1}(B))$.
Then $E \left( e^{it \int fdN} \right) = \exp \left( \int (e^{ity} - 1) \, L(dy) \right)$. Therefore, the characteristic function of $\int fdN$ is exactly the exp($\Psi_{L,0,c}$) where $c = \int \tau(x)L(dx)$.

**Proof of the theorem.**

1. First prove that $e^{\Psi(t)}$ is a characteristic function, and the infinite divisibility is obvious ($n$-th root is $\Psi_{(L/n,\sigma^2/n,c/n)}$). Fix $t$. Observe that for $|x| < 1$
$$e^{itx} - 1 - it\tau(x) = e^{itx} - 1 - itx \leq cx^2t^2, \quad (1)$$
for $|xt|$ small. Therefore, the integral converges because $\int_{-1}^{1} x^2L(dx) < +\infty$ and $L\{(-\varepsilon, \varepsilon)^c\} < +\infty$. Hence $\Psi(t)$ is well-defined complex number for all $t \in \mathbb{R}$.

2. Since the product of characteristic functions is also a characteristic function, assume without loss of generality that $\sigma^2$ and $c$ are both 0. Let $L_n$ be $L$ restricted to $\left( -\frac{1}{n}, \frac{1}{n} \right)^c$. Note that $\exp(\Psi_{L_n,0,0}(t))$ is a characteristic function – since $L_n$ is finite, $\exp(\Psi_{L_n,0,0}(t))$ is the characteristic function of shifted Compound Poisson variable($L_n$). From (1) and the dominated convergence theorem we see
$$\lim_{n \to \infty} \Psi_{L_n,0,0}(t) = \Psi_{L,0,0}(t).$$

Since $\exp$ is continuous function we immediately have that $\exp(\Psi_{L_n,0,0}(t)) \to \exp(\Psi_{L,0,0}(t))$ and it only remains to prove that $\Psi(t)$ is continuous at 0 (in order to apply the Lévy continuity theorem).
4 References