1 Prerequisites

Poisson Process, Probability Spaces, Random Variables

2 Summary

Definition of Poisson Point Process with some examples, notably the Poissonization of the multinomial distribution.

3 Poisson Point Processes

Definition 1 A Poisson Point Process (P.P.P.) with intensity measure $\mu$ on $(\mathcal{S}, \mathcal{S})$ is a collection of random variables $N(B, \omega)$, $B \in \mathcal{S}$, $\omega \in \Omega$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that:

1. $N(B) = N(B, \omega)$, $B \in \mathcal{S}$, $\omega \in \Omega$;

2. $N(\cdot, \omega)$ is a non-negative integer or $\infty$-valued measure on $(\mathcal{S}, \mathcal{S})$ for each $\omega \in \Omega$;

3. $N(B, \cdot)$ is a r.v. with Poisson($\mu(B)$) distribution:

$$\mathbb{P}(N(B) = k) = \frac{e^{-\mu(B)}(\mu(B))^k}{k!} \text{ for all } B \in \mathcal{S};$$

4. If $B_1$, $B_2$, ... are disjoint sets then $N(B_1, \cdot)$, $N(B_2, \cdot)$, ... are independent random variables.

Example 1 Let $S = \mathbb{R}_+$, $\mathcal{S} = \text{Borel}(\mathbb{R}_+)$, $\mu = \lambda \cdot \text{Lebesgue}$. Let $N(B, \omega)$ be the measure of $B$, whose cumulative distribution function is defined as the counting process
(N_t, t \geq 0) for a PP(\lambda) as described before. That is,

\[ N([0, t]) := N_t = \sum_{k=1}^{\infty} 1\{T_k \leq t\} \]

So,

\[ N(B) = \sum_{k=1}^{\infty} 1_{(T_k \in B)} \]

the number of \( T_k \) which fall in \( B \)

**Example 2** \( S = \mathbb{R} \), \( S = \text{Borel}(\mathbb{R}) \), \( \mu = \lambda \cdot \text{Lebesgue} \). Stick together independent \( \text{PP}(\lambda) \) on \( \mathbb{R}_+ \) and \( \mathbb{R}_- \).

**Theorem 1** Such a P.P.P. exists for any \( \sigma \)-finite measure space.

**Proof:** The only convincing argument is to give an explicit construction from sequences of independent random variables. Begin by considering the case \( \mu(S) < \infty \).

1. Take \( X_1, X_2, ... \) to be i.i.d. random variables with form \( \mu(\cdot|S) \) so that \( P(X_i \in B) = \frac{\mu(B)}{\mu(S)} \).
2. Take \( N(S) \) to be a Poisson random variable with mean \( \mu(S) \), independent of the \( X_i \)'s. Assume all random variables are defined on the same probability space \((\Omega, \mathcal{F}, P)\).
3. Define \( N(B) = \sum_{i=1}^{N(S)} 1_{(X_i \in B)} \), for all \( B \in S \).

Note that this \( N(B, \omega) \) is a P.P.P. with intensity \( \mu \). (Use thinning property of Poisson distributions.)

**Example 3** (Poissonization of multinomial) If \( B_1, B_2 \) are disjoint events and \( B_1 \cup B_2 = S \)

\[ P(N(B_1) = n_1, N(B_2) = n_2) = P(N = n_1 + n_2)P(N(B_1) = n_1, N(B_2) = n_2|N = n_1 + n_2) \]

\[ = e^{-\lambda} \frac{\lambda^{n_1+n_2}}{(n_1+n_2)!} \left( \frac{n_1+n_2}{n_1} \right) p^{n_1} q^{n_2}, \]

where \( N = N(S) \), \( p = \frac{\mu(B_1)}{\mu(S)} \), and \( q = \frac{\mu(B_2)}{\mu(S)} \).

4 References