1 Prerequisites

Markov Chains

2 Summary

It is proved that on irreducible, recurrent Markov chains there is positive invariant measure that is unique up to multiplicative constants, and that in the positive recurrent case this measure can be normalized to a probability measure.

3 Stationary Measures on Irreducible, Recurrent Markov Chains

Throughout these notes, we will assume that the Markov chain $X_n$ has a countable state space $S$ and is irreducible and recurrent, unless stated otherwise. As you read these notes, be careful to distinguish measures and probability measures. Recall some definitions and basic results:

- A measure $\mu$ on $S$ is said to be stationary if $\mu P = \mu$, i.e. if
  \[ \sum_{i \in S} \mu(i) P(i, j) = \mu(j) \]

- A state $i$ is recurrent if it satisfies $\mathbb{P}_i(T_i < \infty)$.

- A chain is called irreducible if for all pairs $i, j \in S$ there is an $n$ such that $P^n(i, j) > 0$.

- In an irreducible chain, either all states are recurrent or all states are transient.
If $X_n$ is transient and irreducible, then $\sum_{n} P^n(i, i) < \infty$ for some (and therefore all) state $i$.

The main result in this section is

**Theorem 1** For each state $i \in S$, there is a positive invariant measure $\mu_i(\cdot)$ on $S$ with $\mu_i(i) = 1$, and

$$\mu_i(j) = \mathbb{E}_i[\text{Number of visits to } j \text{ before } T_i] = \mathbb{E}_i \left[ \sum_{n=0}^{T_i-1} 1(X_n = j) \right],$$

where $T_i = \inf\{n \geq 1 : X_n = i\}$. Furthermore, all positive invariant measures are equivalent up to multiplication by constant factors.

**Remark:** The preceding theorem applies to both the positive and the null recurrent cases:

$$\mathbb{E}_b[T_b] = \sum_{j \in S} \mu_b(j) \begin{cases} < \infty \quad \text{Positive recurrent} \\ = \infty \quad \text{Null recurrent} \end{cases}$$

Note, however, that in the null recurrent case, the invariant measure $\mu_i$ cannot be normalized to a probability measure. One further subtle point: if there is a stationary probability measure $\pi$ then $\mu_i$ as defined above can be normalized to a probability measure (which must be $\pi$, by the uniqueness assertion of the theorem), and thus $P$ is positive recurrent.

**Proof:** We will delay the question of uniqueness for the moment. Assume that $\pi$ is nonnegative and invariant with $\pi(i) > 0$ for some $i$. By irreducibility, there is $n$ with $P^n(i, j) > 0$. By invariance, $\pi = \pi P = \pi P^n$, i.e.

$$\pi(j) = \sum_{i \in S} \pi(i) P^n(i, j) > 0$$

This shows that if any state has positive mass under a nonnegative invariant measure, then so do all others. Since $\mu_i(i) = 1$ and $\mu_i$ is nonnegative by definition, positivity of $\mu_i(\cdot)$ will follow from invariance.

To see that $\mu_i$ is invariant, apply definitions:

$$\mu_i(j)P(j, k) = \sum_{n=0}^{\infty} \mathbb{P}_i(T_i > n, X_n = j)P(j, k) = \sum_{n=0}^{\infty} \mathbb{P}_i(T_i > n, X_n = j)\mathbb{P}(X_{n+1} = k | \mathcal{F}_n)$$

$$= \sum_{n=0}^{\infty} \mathbb{P}_i(T_i > n, X_n = j, X_{n+1} = k) = \mathbb{E}_i \left( \sum_{n=0}^{\infty} 1(T_i > n, X_n = j, X_{n+1} = k) \right)$$
There are two cases we need to consider. If \( k \neq i \), then the final expression is the expected number of \( j - k \) pairs before \( T_i \). Sum over \( j \) and use Fubini’s theorem to obtain

\[
\sum_{j \in S} \mu_i(j)P(j, k) = \sum_{j \in S} \mathbb{E}_i \left( \sum_{n=0}^{\infty} 1(T_i > n, X_n = j, X_{n+1} = k) \right) = \mu_i(k)
\]

If \( k = i \), then \( 1(T_i > n, X_n = j, X_{n+1} = k) = 1(T_i = n + 1, X_n = j) \). Sum over \( j \) and use Fubini’s Theorem to obtain

\[
\sum_{j \in S} \mu_i(j)P(j, i) = \sum_{j \in S} \mathbb{E}_i \left( \sum_{n=0}^{\infty} 1(T_i = n + 1, X_n = j) \right) = E_i(1(T_i < \infty)) = 1
\]

the last equality following from recurrence. However, by definition of \( \mu_i \), \( \mu_i(i) = 1 \) (the expected number of visits to \( i \) before the next visit to \( i \) is 1, since we count the visit at time \( n = 0 \)). This concludes the proof that \( \mu_i(\cdot) \) is an invariant measure.

The measure \( \mu_i \) does not assign infinite mass to any point. Assuming otherwise, there is \( j \) with \( \mu_i(j) = \infty \). For any \( k \), by recurrence there is \( n \) with \( P^n(j, k) > 0 \). By invariance \( \mu_i = \mu_iP = \mu_iP^n \), so

\[
\mu_i(k) = \sum_{l \in S} \mu_i(l)P^n(l, k) \geq \mu_i(j)P^n(j, k)
\]

which shows that \( \mu_i(k) = \infty \). In other words, if any point has infinite \( \mu_i \) mass, then so do all points. However, by definition \( \mu_i(i) = 1 \).

Before we address uniqueness, we need a lemma:

**Lemma 1** If \( h \) is nonnegative and harmonic with respect to an irreducible, recurrent Markov chain then \( h \) is constant.

Suppose \( h \) is nonnegative and harmonic, i.e. \( h \geq 0 \) and

\[
h(y) = \sum_{x \in S} h(y)P(x, y)
\]

Then \( (h(X_n))_{n \geq 0} \) is a non-negative martingale. Therefore \( h(X_n) \) converges almost surely to a limit random variable. Since \( X_n \) is an irreducible recurrent Markov chain,

\[
\mathbb{P}_i\{X_n = j \text{ i.o.}\} = 1 \quad \forall i, j, \quad \text{so} \quad \mathbb{P}_i\{h(X_n) = h(j) \text{ i.o.}\} = 1 \quad \forall i, j
\]

Convergence almost surely and \( h(X_n) = h(j) \) infinitely often imply \( h(X_n) = h(j) \) for all \( j \), so \( h \) is constant, i.e. \( h \equiv c \in \mathbb{R} \).
We continue with the uniqueness assertion. The key point is that there exists at least one strictly positive invariant measure \( \mu \): fix an \( i \in S \) and set \( \mu = \mu_i \). Define
\[
\hat{P}(j, i) \overset{\text{def}}{=} \frac{\mu(i)P(i, j)}{\mu(j)}
\]
\( \hat{P} \) is well-defined because \( \mu \) is strictly positive and not infinite, and \( \hat{P} \) is a transition probability matrix because
\[
\sum_{i \in S} \hat{P}(j, i) = \sum_{i \in S} \frac{\mu(i)P(i, j)}{\mu(j)} = 1
\]
The chain governed by \( \hat{P} \) is irreducible and recurrent. To see irreducibility, note that
\[
\hat{P}^2(j, i) = \sum_{k \in S} \hat{P}(j, k)\hat{P}(k, i) = \sum_{k \in S} \frac{\mu(k)P(k, j)}{\mu(j)} \frac{\mu(i)P(i, k)}{\mu(k)} = \frac{\mu(i)}{\mu(j)} \sum_{k \in S} P(i, k)P(k, j)
\]
and by a similar argument, we could show
\[
\hat{P}^n(j, i) = \frac{\mu(i)P^n(i, j)}{\mu(j)}
\]
Since \( \mu > 0 \),
\[
\hat{P}^n(j, i) > 0 \iff P^n(i, j) > 0
\]
which implies \( \hat{P} \) is irreducible iff \( P \) is irreducible.

To see recurrence,
\[
P \text{ is recurrent} \iff \sum_{n=0}^{\infty} P^n(i, i) = \infty \iff \sum_{n=0}^{\infty} \hat{P}^n(i, i) = \infty \iff \hat{P} \text{ is recurrent}
\]
Now, let \( \nu \) be another nonnegative stationary measure. Then:
\[
\sum_i \nu(i)P(i, j) = \nu(j) \Rightarrow \sum_i \nu(i)\hat{P}(j, i)\frac{\mu(j)}{\mu(i)} = \nu(j) \Rightarrow \sum_i \hat{P}(j, i)\frac{\nu(i)}{\mu(i)} = \frac{\nu(j)}{\mu(j)}
\]
Defining \( h(j) := \frac{\nu(j)}{\mu(j)} \), the last equality is \( \hat{P}h = h \) in matrix form, so \( h \) is a nonnegative harmonic function. Since \( \hat{P} \) is irreducible and recurrent, it follows from our lemma that \( h \) is constant, i.e. \( \nu \) is a constant multiple of \( \mu \). \( \blacksquare \)

4 References

Durrett, Section 5.4