1 Prerequisites

Integrability, Convergence in $L^1$, Dominated Convergence Theorem (DCT), Monotone Convergence Theorem (MCT)

2 Summary

Uniform integrability gives necessary and sufficient conditions for a sequence of random variable to converge in $L^1$.

3 Motivation

Suppose that a sequence of random variables $X_n, n \geq 0$, has $X_n$ integrable for each $n$ (i.e. $E[|X_n|] < \infty$) and $X_n \xrightarrow{a.s.} X$ (This discussion is also possible for convergence in probability). We would like to know if $E[|X|] < \infty$ and if $E[X_n] \to E[X]$.

- It is not, in general, true that $E[|X|] < \infty$. For example, take $X$ such that $E[|X|] = \infty$ and define $X_n$ as $X_n = X1_{|X| \leq n}$.

- Even if $E[|X|] < \infty$, $E[X_n]$ does not necessarily converge to $E[X]$. Consider the usual example where $X_n = n1_{A_n}$ with $P(A_n) = 1/n$

Those examples were the bad news, but there is some good news.

- The DCT tells us that if $|X_n| < Y$ and $E[Y] < \infty$, then $E[|X|] < \infty$ and $E[X_n] \to E[X]$.

- The MCT ensures that if $0 \leq X_n \uparrow X$ and $\sup_n E[X_n] < \infty$, then $E[X_n] \uparrow E[X] < \infty$. 
If $X_n$ converges to some $X$ in $L^p$, then $E[|X_n - X|^p] \to 0$. Also, $E[|X_n - X|^{p'}] \to 0$ for $1 \leq p' \leq p$. So if $E[X] < \infty$, we have $E[X_n] \to E[X]$.

These results give sufficient conditions. Uniform integrability gives necessary and sufficient conditions.

### 4 Uniform Integrability

**Definition 1** A collection of random variables $\{X_i, i \in I\}$ (where $I$ is an index set) is said to be **uniformly integrable (UI)** if

$$\lim_{x \to \infty} \sup_i E[|X_i|1_{X_i > x}] = 0 \quad (1)$$

**Note:**

A collection of random variables $\{X_i, i \in I\}$ is said to be **tight** if

$$\lim_{x \to \infty} \sup_i E[1_{X_i > x}] = 0$$

Thus uniform integrability is stronger than tightness.

From a random variable $X \in L^1$ defined on $(\Omega, \mathcal{F}, P)$, a large collection of UI random variables can be built. Let $X_G = E[X|G]$ for any $\sigma$-field $G \in \mathcal{F}$. Let’s verify that $X_G$ is UI.

Consider (w.l.o.g) that $X \geq 0$ (thanks to linearity of UI and the assumption of $L^1$ convergence.) The following result will be used without proof.

**Theorem 1**

$X_i$ is UI $\iff E[X_i 1_A] \to 0$ as $P(A) \to 0$.

Now

$$E[X_G 1_{X_G > x}] = E[X 1_{X_G > x}] \text{ by definition of cond. exp.}$$

$$\to 0 \text{ as } x \to \infty$$

Also

$$P(X_G > x) \leq \frac{E[X_G]}{x} = \frac{E[X]}{x} \to 0 \text{ as } x \to \infty$$

Thus, using the previous theorem with $A \equiv \{X_G > x\}$, we get the uniform integrability of $X_G$.

The following is the main result of uniform integrability. The proof can be found in Durrett Section 4.5.
Theorem 2 Assume that $X_n \overset{a.s.}{\rightarrow} X$ (also works with $\overset{p}{\rightarrow}$). The following are equivalent:

i. $X_n$ is UI

ii. $X_n \overset{L^1}{\rightarrow} X$

iii. $\mathbb{E}[\|X_n\|] \rightarrow \mathbb{E}[\|X\|] < \infty$

5 Reference

Durrett, Rick (2005) Probability: Theory and Examples, 3e, Section 4.5