1 Prerequisites

Stopping Times, Martingales

2 Summary

Definitions and proofs of Wald’s Equations, followed by some examples.

3 Wald’s Equations

3.1 Wald’s Identity

Lemma 1 (Wald’s Identity) Let $X, X_1, X_2, X_3, ...$ be i.i.d. random variables with $\mathbb{E}|X_i| < \infty$ and $T$ be a stopping time for $\{\mathcal{F}_n\}$ where $\mathcal{F}_n = \sigma(X_1, X_2, ..., X_n)$ with $\mathbb{E}(T) < \infty$. Let $S_n = X_1 + X_2 + ... + X_n$. Then $\mathbb{E}S_T = \mathbb{E}X_T \mathbb{E}T$.

Proof:

$$\mathbb{E}S_T = \mathbb{E}\left(\sum_{n=1}^{T} X_n\right) = \mathbb{E}\left(\sum_{n=1}^{\infty} X_n \mathbf{1}_{(T \geq n)}\right) = \sum_{n=1}^{\infty} \mathbb{E}(X_n \mathbf{1}_{(T \geq n)})$$

Notice

$$\begin{align*}
(T \leq n) &\in \mathcal{F}_n \quad n = 0, 1, 2, \ldots \\
\iff (T > n) &\in \mathcal{F}_n \\
\iff (T \geq n + 1) &\in \mathcal{F}_n \\
\iff (T \geq n) &\in \mathcal{F}_{n-1} \quad n = 1, 2, 3, \ldots
\end{align*}$$
Thus, $1_{(T \geq n)} \in \mathcal{F}_{n-1}$ and $X_n$ independent of $\mathcal{F}_{n-1}$, so

$$
\mathbb{E}S_T = \sum_{n=1}^{\infty} \mathbb{E}X_n \mathbb{E}1_{(T > n-1)} = \mathbb{E}X \mathbb{E} \left( \sum_{n=1}^{\infty} 1_{(T \geq n)} \right) = \mathbb{E}X \mathbb{E}T.
$$

\[ \square \]

### 3.2 Wald’s Second Identity

**Lemma 2 (Wald’s Second Identity)** Let $X_1, X_2, \ldots$ be i.i.d. with $\mathbb{E}X_n = 0$ and $\mathbb{E}X_n^2 = \sigma < \infty$. If $T$ is a stopping time with $\mathbb{E}T < \infty$ then $\mathbb{E}S_T^2 = \sigma^2 \mathbb{E}T$.

**Proof:** Recall that $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$. Let $M_n := S_n^2 - n\sigma^2$, where $\sigma^2 = \mathbb{E}[X^2]$. Check that $M_n$ is a martingale:

$$
\mathbb{E}[M_{n+1} | \mathcal{F}_n] = \mathbb{E}[S_{n+1}^2 - (n+1)\sigma^2 | \mathcal{F}_n]
= \mathbb{E}[S_n^2 + 2X_{n+1}S_n + X_{n+1}^2 - (n+1)\sigma^2 | \mathcal{F}_n]
= S_n^2 - n\sigma^2
= M_n.
$$

First, consider the case of a bounded stopping time $T$. We know $0 = \mathbb{E}[M_0] = \mathbb{E}[M_T] = \mathbb{E}[S_T^2 - T\sigma^2]$. Therefore, $\mathbb{E}[S_T^2] = \mathbb{E}[T]\sigma^2$ if $\mathbb{P}(T \leq N) = 1$ for some non-random $N < \infty$.

Now look at the general case when $T$ is unbounded. Consider $T \land n$ instead of $T$. We have $\mathbb{E}[S_{T \land n}^2] = \mathbb{E}[T \land n]\sigma^2$ for every $n = 1, 2, 3, \ldots$. Let $n \to \infty$. Since $T \land n \uparrow T$ as $n$ increases, $\mathbb{E}[T \land n] \uparrow \mathbb{E}[T] < \infty$ (by assumption). Also, $S_{T \land n}$ is a martingale (because $S_n, n = 0, 1, \ldots$ is a martingale). However, since a martingale in $L^2$ has orthogonal increments only diagonal terms add up and hence $\mathbb{E}[S_{T \land n}^2]$ is increasing in $n$. We now argue that $\mathbb{E}[S_{T \land n}^2]$ increases to $\mathbb{E}[S_T^2]$. Since $\mathbb{E}[S_{T \land n}^2] = \mathbb{E}[T \land n]\sigma^2 < \sigma^2 \mathbb{E}[T] < \infty$, we conclude that $S_{T \land n}$ is bounded in $L^2$. Therefore, by boundedness and orthogonality property, $S_{T \land n} \in L^2$ converges to some limit in $L^2$. However, $S_{T \land n} \to S_T$ a.s. Therefore, $\mathbb{E}[S_T^2] = \mathbb{E}[T]\sigma^2$.

\[ \square \]

### 3.3 Examples

It might seem as if $\mathbb{E}X_i$ being the same for all $i$ would suffice for Wald’s (first) identity to hold. More than this is needed, however. $\mathbb{E}|X_i| < \infty$ needs to hold uniformly, without which the summation and integral (in the calculation of expected value of $S_T$) cannot be exchanged. The following example illustrates this.
Example 1 Define $X_i$ as $P(X_i = \pm 2^i) = \frac{1}{2}$. Let $T = \{\inf n : S_n > 1\}$. Clearly, $P(T = n) = \frac{1}{2^n}$, $ET = 2 < \infty$ and $E S_T \geq 1$. However, $EX_i = 0$, which clearly violates Wald’s identity. Note that here $E|X_i| = 2^i \to \infty$, as $i \to \infty$.

The classic *Gambler’s Ruin* formula can be derived using Wald’s identity.

Example 2 (Gambler’s Ruin) The problem is that of a simple, symmetric random walk that starts at $X_0 = 0$, i.e. $X_i$ has a probability of $\frac{1}{2}$ for both 1 and $-1$. Let $a, b > 0$ be two integers. Define $T = \inf\{n|S_n = b \text{ or } S_n = -a\}$. Think about a gambler starting with a net profit of 0 and wondering about the chance she wins $b$ before experiencing a net loss of $a$. Now, $P(S_T = b) = P(T_b < T_{-a})$ where $T_x = \inf\{n|S_n = x\}$. Similarly, $P(S_T = -a) = P(t_{-a} < T_b)$. Using Wald’s identity with $EX = 0$, we have

$$E(S_T) = bP(T_b < T_{-a}) - aP(T_{-a} < T_b) = 0.$$ 

We also have

$$P(T_b < T_{-a}) + P(T_{-a} < T_b) = 1.$$ 

Using these two equations, we find

$$P(T_b < T_{-a}) = \frac{a}{a+b},$$

$$P(T_{-a} < T_b) = \frac{b}{a+b}.$$ 

4 References