1 Prerequisites

Markov chain, stationary distribution.
Strong Markov property.

2 Summary

This topic gives two approaches to calculate the area under the random walk with negative drift.

3 Example of the area under the Random Walk

Consider a random walk on \{0, 1, 2, \ldots\}. Start at 1, there is a probability of \( p \) of going up and \( q \) of going down at each step. (For convenience, let 0 be reflecting i.e. \( p(0, 1) = 1 \). but in the way we will define \( T_0 \), the behavior at 0 will not matter.)

Assume \( q > p \). Let \( T_0 \) be the first time the random walk hits 0 i.e. \( T_0 = \inf\{n \geq 1 : X_n = 0\} \). From example 3.5 of section 5.3 from Durrett, we have \( \mathbb{P}(T_0 < \infty) = 1 \).

**Question:** What is the area under the path until \( T_0 \)? Namely, let \( A \) be the area under the path, we want to compute \( \mathbb{E}_1(A) \). (the subscript “1” means it starts in state 1)

Let’s marshall some resources and try to write down a formula for \( A \) (write \( T \) for \( T_0 \) as shorthand):
\[ A = \frac{X_0 + X_1}{2} + \frac{X_1 + X_2}{2} + \ldots + \frac{X_{T-1} + X_T}{2} \]
\[ = \sum_{n=1}^{\infty} \frac{X_{n-1} + X_n}{2} 1(n < T) \]
\[ = \frac{1}{2} + \sum_{n=1}^{\infty} X_n 1(n < T) \quad \text{(note } \frac{X_T}{2} = 0 \text{ because } X_T = 0 \text{ by definition of } T) \]

We could try and write down a formula for \( \mathbb{E}(X_n 1(n < T)) \) but this is hard!

Differently, let’s look at the sum from another perspective, considering the number of hits to a certain level \( j \). Formally, define \( N_j = |\{n : X_n = j \text{ and } n < T\}| \), number of times state \( j \) is hit before \( T_0 \). Then,

\[ A = \frac{1}{2} + \sum_{j=1}^{\infty} j N_j \]

The advantage of the \( N_j \)’s is that we can compute \( \mathbb{E}(N_j) \) because of the connection between the “expected number of hits in an x-block” and the stationary measure, stated as Theorem 5.4.3 in Chapter 5 of Durrett.

Carry out the computation: define \( \mu(j) = \mathbb{E}_1(N_j) \) for \( j \geq 1 \), and let \( \mu(0) = 1 \). For convenience, define state 0 to be reflecting as in the beginning. Then \( \mu \) is a stationary measure by theorem 5.4.3. Because the random walk occurs on a chain (an infinite chain, a special case of trees, c.f. random walk on graphs in Example 5.4.5 in Durrett), in the stationary case, probability mass has nowhere to escape but to obey the “detailed balance” condition along each edge:

\[ \mu(0) = 1 \]
\[ \mu(j) p = \mu(j+1) q \]
\[ \mu(0) \cdot 1 = \mu(1) q \]

(Detailed balance also implies that the chain is reversible.) Solve for \( \mu(j) \) gives:

\[ \mu(j) = \frac{1}{q} \left( \frac{p}{q} \right)^{j-1} \]

As a check, compute

\[ E_1(T) = \sum_{j=1}^{\infty} \mu(j) = \sum_{j=1}^{\infty} \frac{1}{q} \left( \frac{p}{q} \right)^{j-1} = \frac{1}{q(1 - \frac{p}{q})} = \frac{1}{q - p} \]

which agrees with results obtained via Wald’s equation.
There’s yet an alternative method by a recurrence argument and a clever use of SMP (strong Markov property):

Let $\alpha_k$ be $\mathbb{E}_k(A)$. Consider where the walk goes on the first step: condition on $X_1$ we see that $\alpha_1 = q \cdot \frac{1}{2} + p \cdot (\frac{3}{2} + \alpha_2)$. In general, we have

\[
\begin{align*}
\alpha_0 & = 0 \\
\alpha_1 & = q\left(\frac{1}{2}\right) + p\left(\frac{3}{2} + \alpha_2\right) \\
\alpha_2 & = q\left(\frac{3}{2} + \alpha_1\right) + p\left(\frac{5}{2} + \alpha_3\right) \\
& \vdots
\end{align*}
\]

However, one can break down the area of $\alpha_2$ considering the first time the walk hits state 1. Up till then, the walk behaves the same as starting from 1 and hit 0. After that, it’s yet an IID copy of the walk from 1 to 0. Hence we have

\[
\alpha_2 = 2\alpha_1 + \frac{1}{q - p} \quad \text{(last term is } \mathbb{E}(T)\text{)}
\]

Combined with the previous expression for $\alpha_1$, we solve for two equations in two unknowns and reach the answer:

\[
\alpha_1 = \frac{1}{1 - 2p}\left(\frac{q}{2} + \frac{3p}{2} + \frac{p}{q - p}\right)
\]

### 4 Reference