1 Prerequisites

Borel-Cantelli Lemmas, Convergence of sequence of real numbers, random variables, probability measure.

2 Summary

Almost sure convergence is a measure theoretic driven mode of convergence. In simple terms a sequence of random variables $X_n$ converges almost surely to a random variable $X$ if that the set of $\omega$ such that $X_n(\omega) \not\to X(\omega)$ is a null set. This section explores the definition of almost sure convergence, some properties of almost sure convergence, and some methods for showing almost sure convergence.

3 Almost Sure Convergence

3.1 Definition

Definition 1 (Almost Sure Convergence) We say $X_n \xrightarrow{a.s.} X$ if $X_n(\omega) \to X(\omega)$ for all $\omega \not\in N$, with $\mathbb{P}(N) = 0$, or equivalently $\mathbb{P}(\omega : X_n(\omega) \to X(\omega) \text{ as } n \to \infty) = 1$.

3.2 Preliminaries for the Study of Almost Sure Convergence

Definition 2 Let $q_n$ be some statement, true or false for each $n$. $q_n$ occurs infinitely often or $(q_n \text{ i.o.})$ if for all $n$ there is $m \geq n$ such that $q_m$ is true, and $q_n$ occurs eventually $(q_n \text{ ev.})$ if there exists $n$ such that for all $m \geq n$, $q_m$ is true. Now let $q_n$ depend on $\omega$, giving events

$$A_n = \{\omega : q_n(\omega) \text{ is true}\}.$$
There are now new events,

\[ \{ A_n \text{ i.o.} \} = \{ \omega : \omega \in A_n \text{ i.o.} \} = \bigcap_n \bigcup_{m \geq n} A_m, \]

and

\[ \{ A_n \text{ ev.} \} = \bigcup_n \bigcap_{m \geq n} A_m. \]

In analysis, \( 1_{(A_n \text{ i.o.})} = \lim_{n \to \infty} \sup_{m \geq n} 1_{A_m} \) and \( 1_{(A_n \text{ ev.})} = \lim_{n \to \infty} \inf_{m \geq n} 1_{A_m}. \)

Given a sequence of events \( A_n \) for each \( \omega \in \Omega \), consider \( 1_{A_n(\omega)} \) as a function of \( n \), \( \omega \mapsto (1, 0, 0, 1, \ldots) \).

**Proposition 1 (de Morgan)** \( \{ A_n \text{ i.o.} \}^c = \{ A_n^c \text{ ev.} \} \) and \( \{ A_n \text{ ev.} \}^c = \{ A_n^c \text{ i.o.} \} \)

**Proposition 2** \( X_n \overset{a.s.}{\to} X \iff \forall \epsilon > 0, \ P(|X_n - X| > \epsilon \text{ i.o.}) = 0. \)

**Proof:** \( X_n \to X \iff \forall \epsilon > 0, |X_n - X| < \epsilon \text{ ev.}, \) so

\[ X_n \overset{a.s.}{\to} X \iff \forall \epsilon > 0, P(|X_n - X| \leq \epsilon \text{ ev.}) = 1 \]
\[ \iff \forall \epsilon > 0, P(|X_n - X| > \epsilon \text{ i.o.}) = 0. \]

\[ \blacksquare \]

### 3.3 Properties of Almost Sure Convergence

Because

\[ X_n \to X \text{ a.s.} \iff X_n - X \to 0 \text{ a.s.}, \]

it is enough to prove for the case of convergence to 0.

**Proposition 3** The following are equivalent:

1. \( X_n \overset{a.s.}{\to} 0 \)
2. \( \forall \epsilon > 0, \ P(|X_n| > \epsilon \text{ i.o.}) = 0 \)
3. \( M_n \overset{p}{\to} 0 \) where \( M_n := \sup_{n \leq k} |X_k| \)
4. \( \forall \epsilon_n \downarrow 0 : \ P(|X_n| > \epsilon_n \text{ i.o.}) = 0 \)
Note: “∀” in Proposition 4 cannot be replaced by “∃”. For example, Let $X_n = (1/\sqrt{n})U_n$, where $U_1, U_2, \ldots$ are independent $U[0,1]$.

Take $\epsilon_n = 1/2/\sqrt{n}$. Then, $P(X_n > \epsilon_n) = P(U_n > 1/2) = 1/2$. So, $P(X_n > \epsilon_n \text{ i.o.}) = 1$.

But if we take $\epsilon_n = 1/\sqrt{n}$. Then, $P(X_n > \epsilon_n) = P(U_n > 1) = 0$.

**Proof:** (only for the equivalence of 1 and 3)

Suppose Proposition 1 holds. If $X_n(\omega) \to 0 \text{ a.s.}$, then $\sup_{n \leq k} |X_k(\omega)| \to 0 \text{ a.s.}$ But this implies that $M_n \to 0 \text{ a.s.}$ Thus, $M_n \xrightarrow{p} 0$.

Conversely, if $M_n \downarrow$ as $n \uparrow$, then we know in advance that $M_n$ has a almost-surely-limit in $[0, \infty]$.

**Lemma 1** If $X_n \xrightarrow{p} X$, then there exists a subsequence $n_k$ such that $X_{n_k} \to X \text{ a.s.}$

**Proof:** It is enough to show that there exists $\epsilon_k \downarrow 0$ such that $\sum_k P(|X_{n_k} - X| > \epsilon_k) < \infty$. We can take $\epsilon_k = 1/k$ and choose $n_k$ so that $P(|X_{n_k} - X| > 1/k) \leq 1/2^k$. Then, $\sum_k P(|X_{n_k} - X| > \epsilon_k) < \infty$, and by Borel-Cantelli Lemma I we can conclude that $X_{n_k} \to X \text{ a.s.}$

4 References