1 Prerequisites

Random variables, independence, expected value

2 Summary

The Borel-Cantelli Lemmas are simple, but they are the basic tool for proving almost sure convergence.

3 Borel-Cantelli Lemmas

Theorem 1 (Borel-Cantelli Lemmas) Let \((\Omega, F, \mathbb{P})\) be a probability space and let \(A_n\) be a sequence of events in \(F\). Then,

1. If \(\sum_n \mathbb{P}(A_n) < \infty\), then \(\mathbb{P}(A_n \text{ i.o.}) = 0\).

2. If \(\sum_n \mathbb{P}(A_n) = \infty\) and \(A_n\) are independent, then \(\mathbb{P}(A_n \text{ i.o.}) = 1\).

Proof: \([\text{BCL I}]\)

\[
\mathbb{P}(A_n \text{ i.o.}) = \lim_{m \to \infty} \mathbb{P}(\bigcup_{n \geq m} A_n)
\leq \lim_{m \to \infty} \sum_{n \geq m} \mathbb{P}(A_n) = 0 \quad \text{since} \quad \sum_{i=1}^{\infty} \mathbb{P}(A_n) < \infty.
\]

Proof: \([\text{BCL I, alternative method}]\) Consider a random variable \(N := \sum_1(A_n)\), i.e., the number of events that occur. Then \(\mathbb{E}[N] = \sum_{n=1}^{\infty} \mathbb{P}(A_n)\) by the Monotone
Convergence Theorem, and
\[ \sum_{n=1}^{\infty} \mathbb{P}(A_n) < \infty \implies \mathbb{E}[N] < \infty \]
\[ \implies \mathbb{P}(N < \infty) = 1 \]
\[ \implies \mathbb{P}(N = \infty) = 0 \]
\[ \implies \mathbb{P}(A_n \text{ i.o.}) = 0 \quad \text{because } (N = \infty) \equiv (A_n \text{ i.o.}). \]

**Proof:** [BCL II] Assume that \( \sum \mathbb{P}(A_n) = \infty \) and the \( A_n \)'s are independent. We will show that \( \mathbb{P}(A_n^c \text{ ev.}) = 0. \)

\[
\mathbb{P}(A_n^c \text{ ev.}) = \lim_{n \to \infty} \mathbb{P}(\cap_{m \geq n} A_n^c) = \lim_{n \to \infty} \prod_{m \geq n} \mathbb{P}(A_n^c) = \lim_{n \to \infty} \prod_{m \geq n} (1 - \mathbb{P}(A_m)) \leq \lim_{n \to \infty} \prod_{m \geq n} \exp(-\mathbb{P}(A_m^c)) \]
\[
= \lim_{n \to \infty} \exp\left(-\sum_{m \geq n} \mathbb{P}(A_m^c)\right) = 0
\]

since \( -\sum_{m \geq n} \mathbb{P}(A_m^c) \to \infty \), as \( n \to \infty \)

For (1), use the following fact (due to the independence of \( A_n \):)
\[
\mathbb{P}(\cap_{m \geq n} A_n^c) = \lim_{N \to \infty} \mathbb{P}(\cap_{n \leq m \leq N} A_n^c) = \lim_{N \to \infty} \prod_{n \leq m \leq N} \mathbb{P}(A_m^c) = \prod_{n \leq m} \mathbb{P}(A_m^c).
\]

For (2), \( 1 - x \leq \exp(-x) \) was used. \[\blacksquare\]

Note that the converse of **BCL I** is false (without extra assumptions). For a trivial example, consider \( A_n = (0, 1/n) \) in \((0,1)\). Then, \( \mathbb{P}(A_n) = 1/n, \sum \mathbb{P}(A_n) = \infty \), but \( \mathbb{P}(A_n \text{ i.o.}) = \mathbb{P}(\emptyset) = 0. \)

**Example 1** Consider random walk in \( \mathbb{Z}^d, d = 0, 1, \cdots \) \( S_n = X_1 + \cdots + X_n \), \( n = 0, 1, \cdots \) where \( X_i \) are independent in \( \mathbb{Z}^d \). In the simplest case, each \( X_i \) has uniform distribution on \( 2^d \) possible strings. i.e., if \( d = 3 \), we have \( 2^3 = 8 \) neighbors
\[
\left\{ \begin{array}{c} (+1,+1,+1) \\ \vdots \\ (-1,-1,-1) \end{array} \right\}.
\]

Note that each coordinate of \( S_n \) does a simple coin-tossing walk independently. We can prove that
\[
\mathbb{P}(S_n = 0 \text{ i.o.}) = \begin{cases} 1 & \text{if } d = 1 \text{ or } 2 \quad \text{(recurrent)} \\ 0 & \text{if } d \geq 3 \quad \text{(transient)} \end{cases}
\]
Proof Sketch: (of (3))

Start with $d = 1$, then

$$\mathbb{P}(S_{2n} = 0) = \mathbb{P}(\text{"+" signs and \"-\" signs})$$

$$= \binom{2n}{n} 2^{-2n}$$

$$\sim \frac{c}{\sqrt{n}} \text{ as } n \to \infty.$$ (6)

Noting the facts that $n! \sim \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$, and that $a_n \sim b_n$ iff $\frac{a_n}{b_n} \to 1$ as $n \to \infty$. Note

$$\sum \left(\frac{1}{\sqrt{n}}\right)^d = \infty \quad d = 1, 2$$

$$< \infty \quad d = 3, 4, \ldots$$ (7)

Thus, $\sum_n \mathbb{P}(S_{2n} = 0) = \infty$, and BCL II and (7) together gives (3).

Example 2 (for the case $d = 1$) \{ $S_2 = 0$ \} is the event of ending up back at the origin at step 2 when we started at the origin. $\mathbb{P}(S_2 = 0) = 1/2$. Note:

$$\mathbb{P}(S_{10,000} = 0) \sim \frac{c}{\sqrt{n}} \approx 1/100,$$

$$\mathbb{P}(S_{10,002} = 0) \approx 1/100,$$

$$\mathbb{P}(S_{10,000} = 0, S_{10,002} = 0) = \mathbb{P}(S_{10,000} = 0)\mathbb{P}(S_{10,002} = 0|S_{10,000} = 0) \approx 1/100 \cdot 1/2,$$

The same result holds for the case $d = 2$.

In general,

$$\mathbb{P}(S_{2n} = 0) = \binom{2n}{n}^d \approx \frac{c^d}{n^{d/2}}.$$ For $d = 2$, this is $\sim \frac{c^2}{n}$ which is not summable. Thus, $\mathbb{P}(S_{2n} = 0 \text{ i.o.}) = 1$. For $d \geq 3$, this is $\sim \frac{c^d}{n^{d/2}}$ which is summable. Then, by BCL I, $\mathbb{P}(S_{2n} = 0 \text{ i.o.}) = 0$.

4 References

Durrett, Probability: Theory and Examples (Third Edition), Section 1.6.