1 Prerequisites

Discrete time Markov chain, Poisson Process, exponential distribution, geometric distribution.

2 Summary

In this section, we study how to construct a continuous time Markov chain on a finite state space with the aid of a discrete time Markov chain (on a finite state space), and we study some properties of the continuous time Markov chains that are constructed this way.

3 Finite State Continuous Time Markov Chain

See Durrett Sec 5.6 for the theory of discrete time recurrent Markov Chains with uncountable state space, as developed following Harris. The general idea is to recognize a suitable regenerative structure, like what happens to a discrete time, discrete space Markov chain each time it comes back to a point. Then decompose the path into blocks which are i.i.d. This idea can also be applied to continuous time, discrete space chains.

We now discuss a continuous time, discrete space Markov Chain, with time-homogeneous transition probabilities. Let $S = \text{state space}$. The theory is easiest if $S$ is finite. Some aspects can be extended to $S$ countable. The book of Karlin and Taylor [5], provides details for most of the following discussion. See also [3, 4] for further developments. General theory of countable state space, continuous time chains is very tricky: see Chung [1] and Freedman [2].

Suppose $(X_t, t \geq 0)$ is a process defined on $(\Omega, \mathcal{F}, \mathbb{P})$, with values in $S$ which is finite.
For each $t$, $\omega \to X_t(\omega)$ is a measurable map from $(\Omega, \mathcal{F}) \to S$. Look at the path: $t \to X_t(\omega)$ for fixed $\omega$. In the finite state space case, we expect this path to be almost surely a step function, with only a finite number of jumps in any finite interval of time.

Make the convention that the path is right-continuous with left limits. Such a process has the time-homogeneous Markov property if

- conditionally given $X_t$, the process $(X_s, 0 \leq s \leq t)$ and $(X_u, t \leq u < \infty)$ are independent;
- $(X_{t+v}, v \geq 0|X_t = i)$ is distributed like $(X_v, v \geq 0)|X_0 = i)$.

Introduce the transition matrices $P_t = ||P_t(i, j)||_{i, j}$, where

$$P_t(i, j) = \mathbb{P}(X_{t+s} = j|X_s = i), s \geq 0, i, j \in S$$

The definition of $P_t$ and the time-homogeneous Markov property yield:

- $P_t(i, j) \geq 0$
- $\sum_{j \in S} P_t(i, j) = 1$
- the semi-group property (Chapman-Kolmogorov equation): $P_sP_t = P_{s+t}$

Right-continuous paths make $X_t \to X_0, a.s.$ as $t \to 0^+$, which implies

$$\lim_{t \to 0^+} P_t = I$$

(the identity matrix). Combining with the semigroup property, we know

$$\lim_{r \to t^+} P_r = \lim_{s \to 0^+} P_{t+s} = \lim_{s \to 0^+} P_sP_t = IP_t = P_t.$$ 

Thus, $P_t$ is a right continuous function of $t$. In fact, $P_t$ is not only right continuous but also continuous and even differentiable. Accepting this, we define

$$A := \frac{d}{dt}P_t|_{t=0}.$$ 

The semi-group property easily implies the following backwards equations and forwards equations:

$$\frac{d}{dt}P_t = AP_t = P_tA$$

Hence, there is the representation:

$$P_t = e^{At} = \sum_{n=0}^{\infty} \frac{t^n A^n}{n!} = I + At + A^2t^2/2! + \ldots$$
In particular,
\[ P_t(i, j) = 1_{(i=j)} + A(i, j)t + o(t) \]
Note that \( P_t(i, j) \geq 0 \), so \( A(i, j) \geq 0 \) for \( j \neq i \)

Let’s look at a very instructive example.

### 3.1 An Instructive Example

**Example 1** Fix a discrete time transition matrix \( \hat{P} \), say
\[ \hat{P}(i, j) = \text{probability of a transition from } i \text{ to } j; \quad i, j = 1, 2, ..., N \]
For any initial state \( i \), construct a discrete time Markov chain \( Y_n \) with transition matrix \( \hat{P} \) as
\[ P_i(Y_n = j) = \hat{P}^n(i, j) \]
Associate \( (Y_n) \) with a Poisson process \( (N_t, t > 0) \) with rate \( \lambda \), such that,
\[ P_t(N_t = k) = e^{-\lambda t} (\lambda t)^k / k! \]

Let the “jumps” defined by \( Y_n \) occur at the times of Poisson process. Note that if \( \hat{P}(i, i) > 0 \) then \( P_i(Y_1 = Y_0) > 0 \), and \( \{Y_1 = Y_0\} \) means that jumps occur at, or after, time \( T_1 \), because of the fact that jumps from any state \( i \) to itself will be invisible in continuous time.

Therefore, given a discrete time chain \( Y \), one may construct a continuous time chain \( X \) having \( Y \) as its jump chain; indeed many such chain \( X \) exist. We give a concrete example of this case.

**Example 2** Let the finite state space be \( S\{0, 1\} \), and let the transition matrix of the chain be \( \hat{P}(0, 1) = \hat{P}(0, 0) = \frac{1}{2}, \quad \hat{P}(1, 1) = \hat{P}(1, 0) = \frac{1}{2} \).

Let \( Y_t = \text{state of chain at time } t \).
Define \( X_t := Y_{N_t} \); then, \( X_t \) and \( Y_{N_t} \) have the relationship we described above.

**Claim 1** Whatever the transition matrix \( \hat{P} \) is, and \( \lambda > 0 \) this process \( X \) is a continuous time Markov chain with stationary transition probability
\[ P_t(i, j) = \sum_{n=0}^{\infty} e^{-\lambda t} (\lambda t)^n / n! \hat{P}^n(i, j). \tag{1} \]
Proof of the claim We leave the reader to check the Markov property.

\[ P_t(i, j) = \mathbb{P}(X(t) = j | X(0) = i) \]  

\[ = \sum_{n=0}^{\infty} \mathbb{P}(X(t) = j, N(t) = n | X(0) = i) \]  

\[ = \sum_{n=0}^{\infty} (\lambda t)^n / n! \ e^{-\lambda t} \ \mathbb{P}(Y_n = j | Y_0 = i) \]  

\[ = \sum_{n=0}^{\infty} e^{-\lambda t} \ (\lambda t)^n / n! \ \hat{P}(i, j) \]  

Note that if we compare equation (1) with

\[ P(t) = e^{At} = \sum_{n=0}^{\infty} \frac{t^n A^n}{n!} \]

and note that

\[ e^{-\lambda t} I = \sum_{n=0}^{\infty} \frac{(-\lambda t)^n}{n!} I^n = I e^{-\lambda t}, \]

we have

\[ P_t = e^{\lambda t(\hat{P} - I)}. \]

Thus, \( A = \lambda(\hat{P} - I). \)

Now define \( H_i := \) holding time in state \( i, \) and recall that the interarrival waiting time between a Poisson process follows an exponential distribution. From the discussions above, we see that \( H_i \sim \exp(q_i) \) where \( q_i = (1 - \hat{P}(i, i)) \lambda, \ q_{i,j} = \hat{P}(i, j) \lambda. \)

The holding time in state \( i \) is the sum of a \( \text{geo}(1 - \hat{P}(i, i)) \) number of independent \( \exp(\lambda) \) variables

### 3.2 Generality of construction

Every finite state Markov chain with hold/jump paths has the same distance as that are created by the construction above.
Given

\[
A = \begin{pmatrix}
-q_1 & q_{1,2} & \cdots & q_{1,N} \\
-q_2 & \ddots & \ddots & \ddots \\
\vdots & \ddots & -q_3 & \vdots \\
q_{N,1} & \cdots & -q_N & \ddots \\
\end{pmatrix}
\]

and we do it with any \( \lambda \geq \max_i q_i \).

Now we shortly mention the hazards for the case when \( S \) is countable instead of finite.

The simplest example is the pure birth process.

**Example 3 Pure Birth Process**

*In this case, we have*

\[
q_{i,i+1} = \beta_i; \\
q_i = -\beta_i; \\
and all other \( q_{ij} = 0 \), \( i = 0, 1, 2, \ldots \)
\]

*We can create this process from independent holds.*

*Let \( H_i = \text{holding time in } i \) be \( \exp(\beta_i) \) variables. Then,*

\[
E(H_0 + H_1 + \cdots) = \sum_{n=0}^{\infty} \frac{1}{\beta_i} \text{ since the mean holding time in state } i \text{ is } \frac{1}{\beta_i}.
\]

*We can take \( \beta_i \), say, \( \beta_i = 2^i \), so that this expectation is finite.*

*But the process might still reach \( \infty \) in finite time a.s. This phenomenon is called explosion. There is no unique way to bring the process back from \( \infty \): it could jump back to 0, 1, 2, \ldots according to any probability distribution over these states, then explode again, come back again independently, and so on. Note that the paths of the process can now have infinitely many jumps in finite time. And the way the process comes back from \( \infty \) is not determined by the \( A \) matrix. Much more subtle behavior is possible with countable state space (Markov chains with instantaneous states). This problem is studied in the Boundary theory of Markov chain.***

**References**


