1 Prerequisites

Martingale convergence theorem, stopping time.

2 Summary

This topic gives examples with fair and unfair coin-tossing games.

3 Examples with Coin-Tossing

Example (Fair coin-tossing game). Suppose that you have a dollar in your pocket, and you play a coin-tossing game with a fair coin where you win a dollar if you get heads, and lose a dollar if you get tails. You don’t play at all if you lose all your money. If $S_n$ denotes the amount of money in your pocket at time $n$, it is clear that $S_n$ is a nonnegative martingale, and that $S_n \to 0$ almost surely.

Example (Unfair coin-tossing game). Suppose that we have a biased coin, with probability $p$ of heads, $q$ of tails. Define i.i.d. random variables $X_i$ which equal 1 when the $i^{th}$ coin toss is a head, and equal -1 when the $i^{th}$ coin toss is a tail. Say that $S_0 = a > 0$ where $a$ is an integer, $S_n = S_0 + X_1 + \ldots + X_n$). $S_n$ is not a martingale, but it is possible to find a function $h(x)$ such that $h(S_n)$ is a martingale. Clearly, if $h(S_n)$ is to be a martingale, we must have

$$h(x) = ph(x+1) + qh(x-1),$$

because given $S_n$, $S_{n+1}$ is $S_n + 1$ with probability $p$ and $S_n - 1$ with probability $q$. Try $h(x) = r^x$, for an $r$ to be determined shortly. From the above equation,

$$r^x = pr^{x+1} + qr^{x-1}$$

and

$$r = pr^2 + q.$$
Solving this by the quadratic equation (and recalling that \( p + q = 1 \)), we obtain the solutions \( r = 1, \ r = q/p \); the former solution is trivial, so we use \( h(x) = (q/p)^x \).

Hence, \( M_n = (q/p)^{S_n} \) is a martingale.

Let \( T = \inf\{n: S_n = 0 \text{ or } b\} \) for \( b > a \) an integer. Now \( \Pr(T < \infty) = 1 \) and \( 0 \leq S_{T \wedge n} \leq b \) together imply that \( (q/p)^{S_{T \wedge n}} \) is bounded between \( (q/p)^0 = 1 \) and \( (q/p)^b \) almost surely. Use the dominated convergence theorem to see that

\[
E \left( \frac{q}{p} \right)^{S_T} = \lim_{n \to \infty} E \left( \frac{q}{p} \right)^{S_{T \wedge n}} = \left( \frac{q}{p} \right)^a,
\]

because \( (q/p)^{S_{T \wedge n}} \) is a martingale. Finally, use the equations

\[
\Pr(S_T = b)\left(\frac{q}{p}\right)^b + \Pr(S_T = 0)\left(\frac{q}{p}\right)^0 = ES_T = (q/p)^a
\]

\[
\Pr(S_T = b) + \Pr(S_T = 0) = 1
\]

to see that \( \Pr(S_T = b) = \frac{(q/p)^a - 1}{(q/p)^b - 1} \), and \( \Pr(S_T = 0) = \frac{(q/p)^b - (q/p)^a}{(q/p)^b - 1} \).

4 Reference