Key Word: Finite dimensional distributions, stochastic processes
Synonym: F.D.D.

1 Prerequisite

The definition of stochastic processes

2 Finite dimensional distributions

Given a stochastic processes \( \{X_i, i \in I\} \), we want to study its distributions.

First, look at the one-dimensional distribution. Consider the whole collection of distributions (laws) of individual \( X_i \). Each \( X_i \) has a distribution on state space \((S, \mathcal{S})\), i.e., \( \mathbb{P}_i(A) := \mathbb{P}(X_i \in A), A \in \mathcal{S} \).

Two-dimensional distributions, \( \mathbb{P}_{ij}(B) = \mathbb{P}((X_i, X_j) \in B), B \in \mathcal{S} \times \mathcal{S} \) (product \( \sigma \)-field on \( S \times S \)).

Similarly, we can consider \( k \)-dimension distributions. Say, for \((i_1, i_2, \cdots, i_k) \in I \times I \times \cdots \times I\), \( \mathbb{P}_{i_1, \cdots, i_k}(C) = \mathbb{P}((X_{i_1}, \cdots, X_{i_k}) \in C) \). The collection of all probability distributions on product of \( S \) is called finite dimensional distributions of the stochastic process \( \{X_i, i \in I\} \), as F.D.D.’s for short.

3 Existence of stochastic processes

Given a process \( X \), we can create a collection of F.D.D.’s. Then, what about the converse? If we just write down some family of F.D.D.’s and wonder if there is a stochastic processes \( X \) with these?

Consider joint distributions of two random variables \( (X, Y) \)

- Joint distribution of two random variables \( (X, Y) \).
• Marginal distribution of $X$.  

• Marginal distribution of $Y$.  

There are certainly some connections among the three. Since $\mathbb{P}(X \in A) = \mathbb{P}((X, Y) \in A \times S)$, then Joint $\Rightarrow$ Marginal. This is so-called projective property, or consistency condition.

More generally, given a joint distribution $(X_{i_1}, \cdots, X_{i_k})$, we can drop variables to deduce the $j$-dimensional distributions for every $j \leq k$ and $\{l_1, \cdots, l_j\} \subset \{i_1, \cdots, i_k\}$. There is a necessary condition of consistency of F.D.D.'s.

**Theorem 1 (Kolmogrov’s Consistency)** If $S$ is a nice measurable space, e.g., $S = \mathbb{Z}$, $\mathbb{R}$, $\mathbb{R}^d$, $\mathbb{R}^\infty$, then the converse holds, i.e., given a consistent collection of F.D.D.'s for $X_i$, $i \in I$, $\exists$ a representation of these F.D.D.’s by a stochastic processes.

The construction of the theorem is done on the product space.

**Canonical setup:**

$\Omega = \{\text{functions } \omega : I \to S\}, \quad \omega = (\omega_i, i \in I)$. Define $X_i(\omega) = \omega_i$.

We will use this notation often, especially with $I = \{0, 1, 2, \cdots\}$.

**Remark:** Problems with this formalism:

* It is completely adequate for countable $I$.

* It is restrictive for uncountable $I$.

The problem with the construction of stochastic process on the product spaces where we have an uncountable index set is that events describing some properties of the sample paths are not in general measurable e.g. $\{\omega : i \to X_i(\omega) \text{ is a function with some property } P \}$, like $P=$measurable, continuous, differentiable.

One way to see this is to note that every measurable set in a product $\sigma$-field on this $\Omega$ is determined by some countable collection of indices.

### 4 Reference