1 Prerequisites

Random variables, independence, expected value, almost sure convergence

2 Summary

Kolmogorov’s maximal inequality is a very useful inequality, which will sometimes give better results than Chebyshev’s inequality. It lead to an approach of proving the strong law of large numbers.

3 Kolmogorov’s maximal inequality

Kolmogorov’s maximal inequality: \(X_1, X_2, \ldots, X_n\) be independent with \(E(X_i) = 0\) and \(\sigma_i^2 = E(X_i^2) < \infty\). Define \(S_k = X_1 + X_2 + \cdots + X_n\), then

\[
\mathbb{P}(\max_{1 \leq k \leq n} |S_k| \geq \epsilon) \leq \frac{E(S_n^2)}{\epsilon^2}
\]

Remark: By the same hypotheses, from Chebyshev’s inequality, we can only get

\[
\mathbb{P}(|S_k| \geq \epsilon) \leq \frac{E(S_n^2)}{\epsilon^2}
\]

Proof: Let \(A_k = \{|S_k| \geq \epsilon, \text{ but } |S_j| < \epsilon \text{ for } j < k\}\), which means things are broken down according to the time when \(S_k\) first exceeds \(\epsilon\). Since \(A_k\) are disjoint and

\[
\bigcup_{k=1}^{n} A_k = \{\max_{1 \leq k \leq n} |S_k| \geq \epsilon\}
\]
\[ \mathbb{E}(S_n^2) = \int_\Omega S_n^2 dP \geq \int_{\bigcup_{k=1}^n A_k} S_n^2 dP = \sum_{k=1}^n \int_{A_k} S_n^2 dP \]

but \( S_n^2 = (S_n - S_k)^2 + 2(S_n - S_k)S_n + S_k^2 \),

\[ (S_n - S_k)^2 \geq 0 \implies \int_{A_k} (S_n - S_k)^2 dP \geq 0 \]

and because \( X_i \) are independent, then \( S_k 1_{A_k} \in \sigma(X_1, \ldots, X_k) \) and \( S_n - S_k = X_{k+1} + \cdots + X_n \in \sigma(X_{k+1}, \ldots, X_n) \), so \( S_k 1_{A_k} \) and \( (S_n - S_k) \) are independent, and

\[ \int_{A_k} (S_n - S_k)S_k dP = \int_{\Omega} (S_n - S_k)S_k 1_{A_k} dP = \mathbb{E}((S_n - S_k)S_k 1_{A_k}) = \mathbb{E}(S_n - S_k)\mathbb{E}(S_k 1_{A_k}) \]

\( \mathbb{E}X_i = 0 \), we get \( \mathbb{E}(S_n - S_k) = 0 \). Also, notice \( |S_k| \geq \epsilon \) on \( A_k \)

Finally,

\[ \mathbb{E}(S_n^2) \geq \sum_{k=1}^n \int_{A_k} S_n^2 dP \geq \epsilon^2 \sum_{k=1}^n \mathbb{P}(A_k) = \epsilon^2 \mathbb{P}(\bigcup_{k=1}^n A_k) = \epsilon^2 \max_{1 \leq k \leq n} |S_k| \geq x \]

which completes our proof.

**Remark:** Notice that the condition of \( X_i \) being independent is used only to show \( \mathbb{E}(S_n - S_k)S_k 1_{A_k} = 0 \). So this inequality holds also for any sequence of r.v.’s \((X_1, \ldots, X_n)\) which satisfy

\[ \mathbb{E}((S_n - S_k)S_k 1_{A_k}) = 0 \]

This leads to the definition of martingale. For martingales with the condition \( \mathbb{E}X_i = 0 \), Kolmogorov’s maximal inequality holds.

The condition ”independent and \( \mathbb{E}X_i = 0 \)” can be replaced with a even weaker condition: for any \( n=1,2,\ldots \), \( \mathbb{E}(X_{n+1} 1_{(F_n)}) = 0, \forall F_n \in \sigma(X_1, \ldots, X_n) \).

### 3.1 Convergence of random variables

Let \( \mathcal{F}' = \sigma(X_n, X_{n+1}, \ldots) \) (the future after time \( n \)) be the smallest \( \sigma \)-field with respect to which all \( X_m, (m \geq n) \) are measurable. Let \( \Gamma = \bigcap_n \mathcal{F}' \) be the remote future or tail \( \sigma \)-field.

Intuitively, \( A \in \Gamma \) if and only if changing a finite number of values does not affect the occurrence of the events.

**Example:** Let \( S_n = X_1 + \cdots + X_n \), It is easy to see \( \{\lim_{n \to \infty} S_n \text{exists}\} \in \Gamma \).

Kolmogorov’s 0-1 law:
If \(X_1, X_2, \cdots\), are independent and \(A \in \Gamma\), then \(\mathbb{P}(A) = 0\) or 1. This tells us \(\mathbb{P}(\lim_{n \to \infty} S_n \text{ exists}) = 0\) or 1. But Kolmogorov’s maximal law helps prove this probability is 1 in some special situations.

**Theorem:** Suppose \(X_1, X_2, \cdots\) are independent with \(\mathbb{E}X_n = 0\). Then if \(\sum_{n=1}^{\infty} \text{var}(X_n) < \infty\), then

\[
\mathbb{P}(\lim_{n \to \infty} S_n \text{ exists}) = 1
\]

**Proof:** To prove this, show \(S_n(\omega)\) converges for \(\omega \in \Omega_0\), with \(\mathbb{P}(\Omega_0) = 1\).

From Kolmogorov’s inequality,

\[
\mathbb{P}(\max_{n \leq m \leq N} |S_m - S_n| > \epsilon) \leq \frac{\text{var}(S_N - S_n)}{\epsilon^2} = \epsilon^{-2} \sum_{k=n+1}^{N} \text{var}(X_k)
\]

Let \(M_n := \sup_{m>n} |S_m - S_n|\), then

\[
\mathbb{P}(M_n > \epsilon) \leq \epsilon^{-2} \sum_{k=n+1}^{\infty} \text{var}(X_k) \to 0 \text{ as } n \to \infty
\]

so denote \(\omega_N = \sup_{m,n>N} |S_m - S_n|\), then \(\omega_N \downarrow\) as \(N \uparrow\) and

\[
\mathbb{P}(M_N > \epsilon) \to 0 \text{, as } N \to \infty
\]

so \(\omega_N \to 0\) almost surely. Denote \(\Omega_0 = \{\lim \omega_N = 0\}\). Then \(\forall \omega \in \Omega_0\), \(S_n(\omega)\) is a Cauchy sequence, and hence the limit exists.

This means

\[
\mathbb{P}(\lim_{n \to \infty} S_n \text{ exists}) = 1
\]

\[\square\]

4 References