1 Prerequisites

Martingales, Stopping Times, Upcrossing inequalities

2 Summary

The martingale convergence theorem gives sufficient conditions for a martingale to converge to a (finite) limit. We present some insights why such conditions are sufficient.

3 Martingale Convergence Theorem

First, recall the upcrossing inequality. Suppose that $X_n$, $n \geq 0$ is a nonnegative supermartingale and let $a$ and $b$ be two real numbers such that $X_0 \leq a < b$.

Lemma 1 (Dubin’s inequality) If $X_n$ is a nonnegative supermartingale, then

$$P(U_{a,b} \geq k) \leq \left( \frac{a}{b} \right)^k$$

The lemma tells that this probability is upper bounded by $a/b$ independently of the (gambling) strategy.

Dubin’s inequality implies that

$$\lim_{n \to \infty} P(U_{a,b} \geq n) = 0$$

Or, in other words,

$$P(U_{a,b} < \infty) = 1$$
for all $X_0 \leq a < b$.

The first constraint can be released and conclude that if $X_n, n \geq 0$ is a nonnegative supermartingale and $0 < a < b$, then

$$P(U_{a,b} < \infty) = 1$$

Suppose that $a$ and $b$ are rational, so

$$P[U_{a,b} < \infty, \forall a, b \in \mathbb{Q}] = 1$$

Now consider $\liminf X_n$ and $\limsup X_n$; if $U_{a,b} = \infty$, then

$$\liminf X_n < a < b < \limsup X_n$$

In fact, if there are infinitely many upcrossings, then $X_n$ goes below $a$ and above $b$ infinitely often. So $\inf X_n$ should be less than $a$ and $\sup X_n$ should be bigger than $b$. Hence

$$P(\liminf X_n < a < b < \limsup X_n) \leq P(U_{a,b} = \infty) = 0$$

Thus

$$P(\liminf X_n = \limsup X_n) = 1$$

which implies the following:

**Theorem 1 (Martingale Convergence Theorem)** If $X_n$ is a nonnegative supermartingale, then it converges almost-surely to an almost-surely finite limit $X_\infty$ with

$$\mathbb{E}[X_\infty] \leq \mathbb{E}[X_0]$$

**Proof:** The existence of an almost-sure finite limit has already been shown. In fact, if $P(\liminf X_n = \limsup X_n) = 1$, then $X_n$ converges to a finite limit. We only need to show the last statement: $\mathbb{E}[X_\infty] \leq \mathbb{E}[X_0]$. For that, make use of Fatou’s lemma.

$$\mathbb{E}[\liminf X_n] \leq \liminf \mathbb{E}[X_n] \leq \mathbb{E}[X_0]$$

which gives the result.

A similar result exists for submartingales also. The result is given here, for the proof refer to Durrett or Billingsley.

**Theorem 2** If $X_n$ is a submartingale with $\sup \mathbb{E}[X_n^+] < \infty$ then, as $n \to \infty$, $X_n$ converges a.s. to a limit $X_\infty$ with $\mathbb{E}[|X_\infty|] < \infty$.

4 Reference

Durrett, Rick (2005) *Probability: Theory and Examples, 3e*, Section 4.2

Billingsley, Patrick (1995) *Probability and Measure, 3e*, Section 35