1 Prerequisites

Basic measure theory, random variables.

2 Summary

This is a brief introduction to real random variables, extended real random variables and simple real random variables. Extended random variables enjoy extended cumulative distribution functions, which can be used to construct a compact space of distribution functions. Simple real random variable approximation, working with monotone class theorem, is a typical method in proving many equalities.

3 Real Random Variables

3.1 Checking Measurability

This theorem about checking measurability will save effort by reducing the verification to a smaller class of sets.

Theorem 1 Let $(\Omega, \mathcal{F})$ be a measurable space and $X : \Omega \to S$. If $S$ has the $\sigma$-field $\sigma(\mathcal{A})$ for an arbitrary collection of sets $\mathcal{A}$, then $X$ is measurable iff $(X \in \mathcal{A}) \in \mathcal{F}$ for $A \in \mathcal{A}$. 
**Proof:** We first prove the reverse direction. Since \( \{ X \in A \} = \{ \omega : X(\omega) \in A \} = X^{-1}(A) \), we have

\[
X^{-1}(A^c) = (X^{-1}(A))^c
\]

\[
X^{-1} \left( \bigcup_i A_i \right) = \bigcup_i X^{-1}(A_i)
\]

\[
X^{-1} \left( \bigcap_i A_i \right) = \bigcap_i X^{-1}(A_i)
\]

Thus, \( X^{-1}(\sigma(A)) = \sigma(X^{-1}(A)) \).

To prove the forward direction, note that the collection \( \mathcal{C} \) of subsets of \( S \) given by
\[
\mathcal{C} = \{ B \subset S : X^{-1}(B) \in \mathcal{F} \}
\]
is a \( \sigma \)-field which contains \( \mathcal{A} \) and hence \( \sigma(\mathcal{A}) \) which is the \( \sigma \)-field generated by \( \mathcal{A} \).

Similarly, if \( S \) has the \( \sigma \)-field \( \sigma(Y_i, i \in I) \), \( X \) is measurable iff each \( Y_i \circ X \) is measurable.

**Fact:** The composition of two measurable maps is measurable.

### 3.2 Real Random Variables and Extended Real Random Variables

Let \( S \) be a topological space. The *Borel \( \sigma \)-field* on \( S \), denoted by \( \mathcal{B}(S) \), is the \( \sigma \)-field generated by open subsets of \( S \). If \( f : S \to T \) is a continuous function, then \( f \) is measurable from \((S, \mathcal{B}(S))\) to \((T, \mathcal{B}(T))\) by the previous theorem.

If \((S, S) = (\mathbb{R}, \mathcal{R})\), then some possible choices of \( \mathcal{A} \) are \( \{(-\infty, x] : x \in \mathbb{R}\} \) or \( \{(-\infty, x) : x \in \mathbb{Q}\} \) where \( \mathbb{Q} \) = the rationals.

For the real line \( \mathbb{R} = (-\infty, \infty) \) and extended real line \( \bar{\mathbb{R}} = [-\infty, \infty] \), the Borel \( \sigma \)-fields can be defined as follows.

\[
\mathcal{B}(\mathbb{R}) = \sigma\{(-\infty, x] : x \in \mathbb{R}\}
\]

\[
\mathcal{B}(\bar{\mathbb{R}}) = \sigma\{[-\infty, x] : x \in \bar{\mathbb{R}}\}
\]

**Definition 1 (Real Random Variable)** Let \((\Omega, \mathcal{F})\) be a measurable space. A real random variable (r.r.v.) is a measurable map from \( \Omega \) to \( \mathbb{R} \).

Thus a function \( X \) with range \( \mathbb{R} \) is a r.v. iff \((X \leq x) \in \mathcal{F}\) for all \( x \in \mathbb{R} \) (by theorem 1). Similarly, extended real random variables (e.r.r.v.) can be defined on range \( \bar{\mathbb{R}} \).
Operations on real numbers are performed pointwise on real-valued functions, e.g.,
\[ Z = X + Y \] means \[ Z(\omega) = X(\omega) + Y(\omega) \] for all \( \omega \in \Omega \)
and \( Z = \lim_n Z_n \) means \( Z(\omega) = \lim_n Z_n(\omega) \) for all \( \omega \in \Omega \).

**Notation for real numbers:**
- \( x \vee y = \max(x, y) \)
- \( x \wedge y = \min(x, y) \)
- \( x^+ = x \vee 0 \)
- \( x^- = -(x \wedge 0) \)
- \( |x| = x^+ + x^- \)

**Theorem 2** If \( X_1, X_2, \ldots \) are e.r.r.v.'s on \((\Omega, \mathcal{F})\), then they are closed under all limiting operations, i.e.,
\[ \inf_n X_n, \sup_n X_n, \liminf_n X_n, \limsup_n X_n \]
are also e.r.r.v.

**Proof:** Since the infimum of a sequence is \(< a \) if and only if some term is \(< a \), we have
\[ \left\{ \inf_n X_n < a \right\} = \bigcup_n \{ X_n < a \} \in \mathcal{F} \]
The proof for supremum follows similarly.

For limit inferior of \( X_n \), we have
\[ \liminf_n X_n := \sup_n \{ \inf_m X_m \} \]
Now note that \( Y_n = \inf_{m \geq n} X_m \) is an e.r.r.v. for each \( n \) and so \( \sup_n Y_n \) is also an e.r.r.v. The proof for limit superior follows similarly.

From the above proof we see that
\[ \Omega_0 = \left\{ \omega : \lim_{n \to \infty} X_n \text{ exists} \right\} = \left\{ \omega : \lim_{n \to \infty} \sup X_n - \lim_{n \to \infty} \inf X_n = 0 \right\} \]
is a measurable set. If \( X_n(\omega) \) converges for almost all \( \omega \), i.e., \( \mathbb{P}(\Omega_0) = 1 \), we say that \( X_n \) **converges almost surely** to a limit \( X \) which is defined on \( \Omega_0 \). \( X \) can be defined arbitrarily on \( \Omega \setminus \Omega_0 \), with different authors preferring different conventions.

### 3.3 Simple Real Random Variables

**Definition 2 (Simple Random Variable)** \( X \) is a simple random variable iff \( X \) is a finite linear combination of indicators, i.e., \( X \) can be expressed as \( X(\omega) = \sum_{i=1}^n c_i 1_{A_i}(\omega) \) where \( c_i \in \mathbb{R} \) and \( A_i \in \mathcal{F} \). A simple r.v. can only take finitely many values.
**Theorem 3** Every real r.v. \( X \) is a pointwise limit of a sequence of simple r.v.’s, which can be taken to be increasing if \( X \geq 0 \).

**Proof:** For \( X \geq 0 \) let,

\[
X_n = \begin{cases} 
\frac{k-1}{2^n} & \text{on } \{ \frac{k-1}{2^n} \leq X < \frac{k}{2^n} \}, 0 \leq k \leq n2^n \\
0 & \text{on } \{ X \geq n \}
\end{cases}
\]

Then \( X_n \uparrow X \). For general \( X \) use the decomposition \( X = X^+ - X^- \). ■

**Corollary 1** Let \( X \) and \( Y \) be real valued r.v.’s. Then so are \( XY, X + Y, X - Y, \min(X,Y), \max(X,Y) \).

**Proof:** Consider \( X_n \uparrow X \) and \( Y_n \uparrow Y \). This implies \( X_n Y_n \uparrow XY \). Similarly, use the previous theorem to pass from simple case to the more general cases. ■

## 4 References