1 Prerequisites

- Discrete time, countable state space Markov chains
- Random walks

2 Summary

These notes cover renewal theory for discrete and non-arithmetic increment distributions. The age process, stationary renewal processes, and the delay distribution are introduced, and Blackwell’s renewal theorem and renewal equations are touched upon.

3 Discrete Renewal Theory

3.1 Preliminaries

We consider a random walk with strictly positive i.i.d. increments. Suppose the increment distribution is the distribution $F$ of the random variable $X$ on the positive integers $\mathbb{Z}^+$. Throughout, we assume $\mu = \mathbb{E}X < \infty$. For a sequence of i.i.d. random variables $X_1, X_2, ...$ with the same distribution as $X$, let

\[
\begin{align*}
    f_n &= \mathbb{P}(X = n) \quad \text{for } n = 1, 2, 3, ... \\
    S_0 &= 0 \\
    S_n &= X_1 + X_2 + ... + X_n
\end{align*}
\]

The “time” $S_n$ is the $n^{\text{th}}$ renewal time. Define the renewal indicator

\[
Z_n = 1(\text{renewal at time } n) = 1(S_k \text{ has value } n \text{ for some } k) = \sum_k 1(S_k = n)
\]
We are interested in
\[ u_n = \mathbb{E}(Z_n) = \mathbb{P}(\text{renewal at time } n) \]
\[ = \sum_{k=0}^{n} f_n^* \]
where \( f_n^* \) is 1 convolved with \( f_k \) times i.e. \( f_n^0 = 1 \) and \( f_n^k = \sum_{j=1}^{n-1} f_j^{(k-1)} f_{n-j} \).

### 3.2 Discrete Renewal Theorem

**Theorem 1** If \((f_n)\) is aperiodic then \( u_n \to 1/\mu \) as \( n \to \infty \) where \( \mu = \mathbb{E}X \)

**Proof:** We prove this by constructing an aperiodic, irreducible Markov chain where the \( k^{th} \) recurrence time for state 0 that is equal to the \( k^{th} \) renewal time. We may then use the limit theorem from Markov chain theory regarding convergence of transition probabilities to the stationary distribution, \( p^n(0,0) \to \pi_0 \) as \( n \to \infty \).

Define the age process as follows:
\[ A_n = n - \max\{S_k : s_k \leq n\} = n - S_{N(n)} \]
where \( N(n) = \sum_{k=1}^{n-1} Z_k = \# \text{ renewals in } [0,n] \). \( A_n \) is the “age” since the last renewal \( S_{N(n)} \). We must check that \( A_n \) is a Markov chain. First note that \( A_{n+1} \) must either be \( A_n + 1 \) or 0. The transition probabilities are then given by
\[ \mathbb{P}(A_{n+1} = 0 | A_n = i, ...) = \frac{f_{i+1}}{\sum_{k=i+2}^{\infty} f_k} = \frac{\mathbb{P}(X = i + 1)}{\mathbb{P}(X > i)} \]
\[ \mathbb{P}(A_{n+1} = i + 1 | A_n = i, ...) = \frac{\sum_{k=i+2}^{\infty} f_k}{\sum_{k=i+1}^{\infty} f_k} = \frac{\mathbb{P}(X > i + 1)}{\mathbb{P}(X > i)} \]

We see that the transition probabilities do not depend on any values of the age process before \( A_n \), and the age process is a Markov chain. Note that the formulas for the transition probabilities are nearly irrelevant since we are only concerned with renewal times. Since \( (A_n = 0) \Leftrightarrow (\text{renewal at time } n) \) and \( A_0 = 0 \), we are only concerned with recurrence times for state 0, and we obtain
\[ u_n = p^n(0,0) \]
where \( p \) is the transition matrix for the age process.
To apply the appropriate Markov chain limit theorems we need to check irreducibility and aperiodicity. Irreducibility is just a property of the state space we choose. Let the state space

\[ S = \begin{cases} 
\{1, 2, 3, \ldots\} & \text{if } \sup\{n : f_n > 0\} = \infty \\
\{1, 2, \ldots, N - 1\} & \text{if } \sup\{n : f_n > 0\} = N 
\end{cases} \]

Aperiodicity is assumed by hypothesis. Thus we obtain

\[ u_n = p^n(0, 0) \to \pi_0 = 1/\mu \]

where \( \pi_0 \) is the probability of being in state 0 in the stationary distribution.

### 3.3 Asymptotic distribution of the age process

From Markov chain theory, we have the following

\[ P(A_n = 0) = p^n(0, 0) \to \pi_0 1/\mu \]
\[ P(A_n = 1) = p^n(0, 1) \to \pi_1 1/\mu \]
\[ P(A_n = 2) = p^n(0, 2) \to \pi_1 2/\mu \]

... and

\[ \mathbb{E}X = \sum_{n=0}^{\infty} P(X > n) \]

which suggests

\[ P(A_n = k) = P(X > k)/\mu \]

This is, in fact, the case and may be verified by solving the system of equations \( \pi P = \pi \) and using the transition probabilities \( P(A_{n+1} = i + 1 | A_n) = P(X > i + 1)/P(X > i) \).

### 4 Non-lattice Case

#### 4.1 Preliminaries

We now consider the case where the increment distribution of a random variable \( X \) is not concentrated on a lattice \( \{\delta, 2\delta, \ldots\} \). In this case, the distribution function \( F \) of \( X \) is called nonarithmentic. Assume that \( \mu = \mathbb{E}X < \infty \). Define the age process in the same way as in the discrete case, \( A_t = t - \max\{S_k : S_k \leq t\} = t - S_{N_t-1} \) where
\( N_t = (\# \text{ renewals in } (0, t]) = \max\{k : S_k \leq t\} - 1 \). For an arbitrary Borel set \( I \), define \( N \) to be the random measure

\[
N(I) = (\# \text{ renewals in } I) = \sum_{k=0}^{\infty} 1(S_k \in I)
\]

\( N \) is called the occupation measure. Similarly, define the renewal measure

\[
U(I) = \mathbb{E}N(I) = \sum_{k=0}^{\infty} \mathbb{P}(S_k \in I)
\]

As in the discrete case we wish to consider the asymptotic distribution of the age process. We start with an example.

**Example 1** Suppose \( X \sim \exp(\lambda) \). Then \( (N_t, t \geq 0) \) is a Poisson process and \( \mathbb{P}(N_t = k) = e^{-\lambda t} \left( \frac{\lambda t^k}{k!} \right) \). Consider the age process.

\[
P(A_t > x) = \mathbb{P}(\text{ no points lie in } (t - x, t)) = e^{-\lambda x} \quad \text{ if } 0 < x < t
\]

\[
P(A_t = t) = e^{-\lambda t}
\]

This gives \( \lim_{t \to \infty} \mathbb{P}(A_t > x) = e^{-\lambda x} \) so \( A_t \xrightarrow{d} X \). and the limit distribution of the age process is the same of the increment distribution. This is unusual. We now consider the general asymptotic distribution of \( A_t \).

Recall that

\[
X = \int_{0}^{X} 1dx = \int_{0}^{\infty} 1(X > x)dx
\]

\[
\mathbb{E}X = \int_{0}^{\infty} \mathbb{P}(X > x)dx
\]

From the discussion of the discrete case where we found that \( \mathbb{P}(A_t = k) \rightarrow P(X > k) \), we may guess that if an asymptotic distribution exists and \( A_t \xrightarrow{d} A_\infty \), then the distribution of \( A_\infty \) is given by

\[
\mathbb{P}(A_\infty \in dx) = \frac{P(X > x)}{\mu} dx
\]

One sees that this is indeed the case after working out exercise 4.5 in section 3.4 of Durrett.
We may also consider the age process backwards. Define the residual lifetime \( R_t \) process by \( R_t = S_{N_t} - t \). By similar arguments to those used for \( A_t \), \( R_t \stackrel{d}{\rightarrow} A_\infty \). By the definition of the age and residual lifetimes, we see that \( A_t + R_t = \) (the value of the increment \( X \) covering \( t \)) \( \stackrel{d}{\rightarrow} X_s \) for some \( X_s \). By symmetry, we may guess that \( (A_t, R_t) \stackrel{d}{\rightarrow} (X_s U, X_u (1 - U)) \) where \( U \sim \text{unif}(0,1) \), and \( U \) is independent of \( X_s \).

### 4.2 Stationary Renewal Processes

We have relied on heuristic arguments thus far. The stationary renewal process allows us to formalize the results.

The idea for creating a stationary renewal process is this. For some renewal process and time point \( t \), we may consider the process to the right of \( t \). This process starts with a delay \( D_t = \min\{S_n: S_n \geq t\} - t \), and after \( D_t \) the process is the same as one started from 0. This renewal process has increments \( X_1, X_2, \ldots \) with \( X_1 \sim G \) for some delay distribution \( G \) and \( X_2, X_3, \ldots \) i.i.d.

For appropriate \( G \), the resulting process \( (N_t, t \geq 0) \) has stationary increments ie. \( N_t - N_s \stackrel{d}{=} N_{t-s} \) for all \( s \leq t \).

**Theorem 2** If the increment distribution \( F \) is nonarithmetic, then there exists a unique delay distribution \( G \) such that the process \( (N_t, t \geq 0) \) has stationary increments. Furthermore, \( G \) is given by

\[
G(t) = \frac{1}{\mu} \int_0^\infty (1 - F(y))dy
\]

and the renewal measure \( U = G + G \ast F + G \ast F \ast F + \ldots = \lambda/\mu \) where \( \ast \) denotes convolution and \( \lambda \) is Lesbesgue measure.

See Durrett section 3.4 or Kallenberg proposition 9.18 for proof details. Note that, in the example, the increment distribution and delay distribution are the same, \( F = G \). Using this result we obtain the following theorem.

**Theorem 3 (Blackwell’s renewal theorem)** A renewal process with increments having a nonarithmetic distribution \( F \) has renewal measure \( U([t, t+h)) \rightarrow h/\mu \).

### 4.3 Renewal equations

In theorem 2, the renewal measure \( U = G + U \ast F \). This leads us to consider the renewal equation, \( H = h + H \ast F \). For the renewal equation, we have the following theorems stated in Durrett.
Theorem 4 If $h$ is bounded, then $H(t) = \int_0^t h(t - s)dU(s)$ is the unique solution to the renewal equation that is bounded on bounded intervals.

Theorem 5 (Renewal theorem) If $F$ is nonarithmetic and $h$ is directly Riemann integrable then as $t \to \infty$

$$H(t) \to \int_0^\infty h(s)ds$$

5 References

1. Durrett, Rick (2005) Probability: Theory and Examples, 3e, Section 3.4