1 Prerequisites

Conditional Expectation

2 Summary

This follows up the section on Conditional Expectation. It links the measure theoretical definitions back to the “undergraduate” probability definitions.

3 Relation to Undergraduate Probability

If \((X, Y)\) has joint density \(f(x, y)\) with respect to Lebesgue measure \(dx \, dy\),

\[
P(X \in dx, Y \in dy) = f(x, y) \, dx \, dy.
\]

This means \(E(g(X, Y)) = \int_{\mathbb{R}} \int_{\mathbb{R}} g(x, y) f(x, y) \, dx \, dy\) for all \(g \geq 0\) or \(g\) bounded. Let the density of \(X\) be \(P(X \in dx) = f_X(x) \, dx\) and the density of \(Y\) be \(P(Y \in dy) = f_Y(y) \, dy\). This means that \(f_X(x) = \int f(x, y) \, dy\) and \(f_Y(y) = \int f(x, y) \, dx\).

Now for all \(y\) with \(f_Y(y) > 0\), define:

\[
f_{X|Y=y}(x) = \frac{f(x, y)}{f_Y(y)} \geq 0, \quad \text{with} \quad \int f_{X|Y=y}(x) \, dx = 1
\]

This is the “formal” conditional density of \(X\) given \(Y = y\). This is related to \(E(X|Y) = \phi(Y)\) by defining \(\phi(y)\) as

\[
\phi(y) = \begin{cases} 
\int x \frac{f(x, y)}{f_Y(y)} \, dx & \text{if } f_Y(y) > 0 \\
0 & \text{if } f_Y(y) = 0
\end{cases}
\]
Check that this is measurable with respect to $\sigma(Y)$: this follows from part of Fubini’s theorem. Also, check that $\mathbb{E}(\phi(Y)h(Y)) = \mathbb{E}(Xh(Y))$ for $h$ measurable.

In the same setting, let $\mathbb{E}(k(X)|Y) = \phi_k(Y)$. $\phi_k(y)$ is defined by simply replacing $x$ with $k(x)$:

$$\phi_k(y) = \begin{cases} \int k(x) \frac{f(x, y)}{f_Y(y)} dy & \text{if } f_Y(y) > 0 \\ 0 & \text{if } f_Y(y) = 0 \end{cases}.$$ 

In this setup, the expectation of any $k(X)$ given $Y$ can be obtained by integration with respect to a regular conditional distribution, which is a distribution of $X$ that depends on the value of $Y$. Here, given $Y = y$, we have the density $f_{X|Y=y}(x) = f(x, y)/f_Y(y)$. 